

An arithmetic Lefschetz-Riemann-Roch theorem

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Abstract. In this article, we consider regular arithmetic schemes in the context of Arakelov geometry, endowed with an action of the diagonalisable group scheme associated to a finite cyclic group. For any equivariant and proper morphism of such arithmetic schemes, which is smooth over the generic fibre, we define a direct image map between corresponding higher equivariant arithmetic K-groups and we discuss its transitivity property. Then we use the localization sequence of higher arithmetic K-groups and the higher arithmetic concentration theorem developed in [T3] to prove an arithmetic Lefschetz-Riemann-Roch theorem. This theorem can be viewed as a generalization, to the higher equivariant arithmetic K-theory, of the fixed point formula of Lefschetz type proved by K. Köhler and D. Roessler in [KR1].

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1 Introduction

The aim of this article is to prove an arithmetic Riemann-Roch theorem of Lefschetz type for the higher equivariant arithmetic K-theory of regular arithmetic schemes in the context of Arakelov geometry. This theorem is an arithmetic analogue of a special case of K ock's Lefschetz theorem in higher equivariant K-theory (cf. [Ko1]), and it also generalizes K ohler-Roessler's Lefschetz fixed point formula [KR1, Theorem 4.4] to the case where higher arithmetic K-groups are concerned. To make things more explicit, let us first recall the study of such Lefschetz-Riemann-Roch problems.

Let X be a smooth projective variety over an algebraically closed field k , and suppose that X is endowed with an action of a cyclic group $\langle g \rangle$ whose order is prime to the characteristic of k . An equivariant coherent sheaf on X is a coherent \mathcal{O}_X -module F on X together with an automorphism $\varphi : g^*F \rightarrow F$. Then the classical Lefschetz trace formula is to give an expression of the alternating sum of the trace of $H^i(\varphi)$ on the cohomology space $H^i(X, F)$, as a sum of the contributions from the components of the fixed point subvariety X_g . For $k = \mathbb{C}$, the field of complex numbers, such a Lefschetz trace formula was presented via index theory and topological K-theory in [ASe, III]. While for general k , a Grothendieck type generalization to the scheme theoretic algebraic geometry is very natural to expect. Precisely, denote by $K_0(X, g)$ the Grothendieck group of the category of equivariant locally free coherent sheaves on X , then $K_0(\text{Pt}, g)$ is isomorphic to the group ring $\mathbb{Z}[k]$ and $K_0(X, g)$ has a natural $K_0(\text{Pt}, g)$ -algebra structure (Pt stands for the point $\text{Spec}(k)$). Let Y be another $\langle g \rangle$ -equivariant smooth projective variety, let $f : X \rightarrow Y$ be a projective morphism compatible with both $\langle g \rangle$ -actions on X and on Y , then we have a direct image map $f_* : K_0(X, g) \rightarrow K_0(Y, g)$ given by

$$E \mapsto \sum_{i \geq 0} R^i f_*(E).$$

Unsurprisingly, the direct image map f_* doesn't commute with the restriction map $\tau : K_0(\cdot, g) \rightarrow K_0((\cdot)_g, g)$ from the equivariant K_0 -group of an equivariant variety to the equivariant K_0 -group of its fixed point subvariety. Namely, the restriction map τ is not a natural transformation between the covariant functors $K_0(\cdot, g)$ and $K_0((\cdot)_g, g)$. Like the other Riemann-Roch problems, the Lefschetz-Riemann-Roch theorem makes a correction of τ such that it becomes a natural transformation. In fact, for any $\langle g \rangle$ -equivariant smooth projective variety X , let N_{X/X_g} stand for the normal bundle associated to the regular immersion $X_g \hookrightarrow X$ and let $\lambda_{-1}(N_{X/X_g}^\vee)$ be the alternating sum $\sum (-1)^j \wedge^j N_{X/X_g}^\vee$, then $\lambda_{-1}(N_{X/X_g}^\vee)$ is an invertible element in $K_0(X_g, g) \otimes_{\mathbb{Z}[k]} \mathcal{R}$ where \mathcal{R} is any $\mathbb{Z}[k]$ -algebra in which $1 - \zeta$ is invertible for each non-trivial n -th root of unity $\zeta \in k$. We formally define $L_X : K_0(X, g) \rightarrow K_0(X_g, g) \otimes_{\mathbb{Z}[k]} \mathcal{R}$ as $\lambda_{-1}^{-1}(N_{X/X_g}^\vee) \cdot \tau$, the Lefschetz-Riemann-Roch theorem reads: the following diagram

$$\begin{array}{ccc} K_0(X, g) & \xrightarrow{L_X} & K_0(X_g, g) \otimes_{\mathbb{Z}[k]} \mathcal{R} \\ \downarrow f_* & & \downarrow f_{g*} \\ K_0(Y, g) & \xrightarrow{L_Y} & K_0(Y_g, g) \otimes_{\mathbb{Z}[k]} \mathcal{R} \end{array} \quad (1)$$

is commutative.

This commutative diagram (1) was presented by P. Donovan in [Do], and later it was generalized to singular varieties by P. Baum, W. Fulton and G. Quart in [BFQ]. The reasoning in the first paper runs similarly to the technic used in Borel-Serre's paper [BS], while the reasoning in the second paper relies on the deformation to the normal cone construction. These two processes are both traditional for producing the Grothendieck type Rieamnn-Roch theorem.

After Quillen and other mathematicians' work, algebraic K-groups are extended to higher degrees and the higher (equivariant) algebraic K-groups of X are defined as the higher homotopy groups of the K-theory space associated to the category of (equivariant) locally free coherent sheaves on X . There are many methods to construct this "K-theory space", but no matter which construction we choose, the tensor product of locally free coherent sheaves always induces a graded ring structure on $K_\bullet(X, g)$. In particular, each $K_m(X, g)$ is a $K_0(X, g)$ -module. Moreover, the functor $K_\bullet(\cdot, g)$ is again covariant with respect to equivariant proper morphisms. Then, for any $m \geq 1$, the following diagram for higher algebraic K-groups which is similar to (1) does make sense:

$$\begin{array}{ccc} K_m(X, g) & \xrightarrow{L_X} & K_m(X_g, g) \otimes_{\mathbb{Z}[k]} \mathcal{R} \\ \downarrow f_* & & \downarrow f_{g*} \\ K_m(Y, g) & \xrightarrow{L_Y} & K_m(Y_g, g) \otimes_{\mathbb{Z}[k]} \mathcal{R}. \end{array} \quad (2)$$

The commutativity of diagram (2), which is named the Lefschetz-Riemann-Roch theorem for higher equivariant algebraic K-theory, was proved by B. Köck in [Kö1]. The main guarantee is an excess intersection formula whose proof also relies on the deformation to the normal cone construction. Moreover, it's worth indicating that the commutative diagram (2), combined with the Gillet's Riemann-Roch theorem for higher algebraic K-theory (cf. [Gi]), deduces a higher Lefschetz trace formula.

In the field of arithmetic geometry, one considers those noetherian and separated schemes $f : X \rightarrow \text{Spec} \mathbb{Z}$ over the ring of integers (actually over even slightly more general rings). In this context, an analogue of the commutative diagram (1) is possible to produce, by endowing X with an action of the diagonalisable group scheme $\mu_n = \text{Spec}(\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}])$ of n -th roots of unity rather than with the action of an automorphism of order n . The reason for this choice is that the fixed point subscheme X_{μ_n} of a regular scheme X equipped with an action of μ_n is still regular and the natural inclusion $i_X : X_{\mu_n} \hookrightarrow X$ is a regular immersion, while the fixed point subscheme of a regular scheme under an automorphism of order n can be very singular over the fibres lying above the primes dividing n . Under this situation, Baum-Fulton-Quart's method still works, so that the commutative diagram (1) holds for μ_n -equivariant schemes over \mathbb{Z} .

In [Th], R. W. Thomason used another way to do the same thing and he even got a generalization of the commutative diagram (2). Thomason's strategy was to use Quillen's localization sequence for higher equivariant algebraic K-groups to show a concentration theorem. This theorem states that, after a suitable localization, the equivariant algebraic K-group $K_m(X_{\mu_n}, \mu_n)_\rho$ is isomorphic to $K_m(X, \mu_n)_\rho$ for any $m \geq 0$, and the inverse map is exactly given by $\lambda_{-1}^{-1}(N_{X/X_{\mu_n}}^\vee) \cdot i_X^*$.

Here, ρ is any prime ideal in $R(\mu_n) := K_0(\text{Spec}\mathbb{Z}, \mu_n) \cong \mathbb{Z}[T]/(1 - T^n)$ which doesn't contain the elements $1 - T^k$ for $k = 1, \dots, n-1$. Then the Lefschetz-Riemann-Roch theorem for μ_n -equivariant schemes

$$\begin{array}{ccc} K_m(X, \mu_n) & \xrightarrow{L_X} & K_m(X_{\mu_n}, \mu_n)_\rho \\ \downarrow f_* & & \downarrow f_{\mu_n*} \\ K_m(Y, \mu_n) & \xrightarrow{L_Y} & K_m(Y_{\mu_n}, \mu_n)_\rho \end{array} \quad (3)$$

follows from the covariant functoriality of $K_\bullet(\cdot, \mu_n)$ with respect to proper morphisms.

Now, let us turn to Arakelov geometry. Let X be an arithmetic scheme over a regular arithmetic ring (D, Σ, F_∞) in the sense of Gillet-Soulé (cf. [GS1]), then X is quasi-projective over D with smooth generic fibre. We denote $\mu_n := \text{Spec}(D[\mathbb{Z}/n\mathbb{Z}])$ the diagonalisable group scheme over D associated to a cyclic group $\mathbb{Z}/n\mathbb{Z}$. By saying X is μ_n -equivariant, we understand that X is endowed with a projective μ_n -action. That means X is projective and there exists a very ample invertible μ_n -sheaf on X . For simplicity, in this article, all μ_n -equivariant arithmetic schemes are supposed to be regular.

For each μ_n -equivariant arithmetic scheme X , K. Köhler and D. Rössler have defined an equivariant arithmetic K_0 -group $\widehat{K}_0(X, \mu_n)$ in [KR1]. This arithmetic K_0 -group is a modified Grothendieck group of the category of equivariant hermitian vector bundles on X , it contains some smooth form class on $X_{\mu_n}(\mathbb{C})$ as analytic datum. The same as the algebraic K_0 -group $K_0(X, \mu_n)$, $\widehat{K}_0(X, \mu_n)$ has a ring structure and it is an $R(\mu_n)$ -algebra. Moreover, direct image maps between equivariant arithmetic K_0 -groups can be defined for equivariant and proper morphism which is smooth over generic fibre, by using Bismut-Köhler-Ma's analytic torsion forms. Choose a Kähler metric for $X(\mathbb{C})$, and let $\overline{N}_{X/X_{\mu_n}}$ be the normal bundle endowed with the quotient metric, then the main theorem in [KR1] reads: the element $\lambda_{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)$ is a unit in $\widehat{K}_0(X_{\mu_n}, \mu_n)_\rho$ and the following diagram

$$\begin{array}{ccc} \widehat{K}_0(X, \mu_n) & \xrightarrow{\Lambda_R \cdot \tau} & \widehat{K}_0(X_{\mu_n}, \mu_n)_\rho \\ \downarrow f_* & & \downarrow f_{\mu_n*} \\ \widehat{K}_0(D, \mu_n) & \longrightarrow & \widehat{K}_0(D, \mu_n)_\rho \end{array} \quad (4)$$

is commutative, where Λ_R is defined as $(1 - R_g(N_{X/X_{\mu_n}})) \cdot \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)$ and $R_g(\cdot)$ is the equivariant R -genus due to Bismut (see below).

Later, two refinements of (4) where D is replaced by a general μ_n -equivariant scheme Y and X is allowed to have singularities on its finite fibres were presented by us in [T1] and in [T2] respectively. The aim of this article is to show an arakelovian analogue of a special case of (3), in which the higher equivariant algebraic K -groups are replaced by the higher equivariant arithmetic K -groups. Hence, our work is a generalization of Köhler-Roessler's Lefschetz fixed point formula to the higher equivariant arithmetic K -theory.

Let us describe our main result a little precise. Firstly, notice that we have constructed a group endomorphism $\otimes \lambda_{-1}(\overline{N}_{X/X_{\mu_n}}^\vee) : \widehat{K}_m(X_{\mu_n}, \mu_n) \rightarrow \widehat{K}_m(X_{\mu_n}, \mu_n)$ and its formal inverse $\otimes \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee) : \widehat{K}_m(X_{\mu_n}, \mu_n)_\rho \rightarrow \widehat{K}_m(X_{\mu_n}, \mu_n)_\rho$ in [T3, Section 5]. In this article, we shall further construct a group endomorphism $R_g(\overline{N}_{X/X_{\mu_n}}) : \widehat{K}_m(X_{\mu_n}, \mu_n) \rightarrow \widehat{K}_m(X_{\mu_n}, \mu_n)$ and we shall prove that this endomorphism $R_g(\overline{N}_{X/X_{\mu_n}})$ is independent of the choice of the metric over $N_{X/X_{\mu_n}}$ after tensoring by \mathbb{Q} . So the expression $\Lambda_R = (1 - R_g(N_{X/X_{\mu_n}})) \cdot \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)$ still makes sense as an endomorphism of $\widehat{K}_m(X_{\mu_n}, \mu_n)_\rho \otimes \mathbb{Q}$. Moreover, for any equivariant and proper morphism $f : X \rightarrow Y$ between μ_n -equivariant arithmetic schemes, which is smooth over generic fibre, we shall prove that there exists a reasonable direct image map $f_* : \widehat{K}_m(X, \mu_n) \rightarrow \widehat{K}_m(Y, \mu_n)$ with $m \geq 1$ and we discuss the transitivity property of the direct image maps up to torsion. Assume that the μ_n -action on Y is trivial, our main theorem reads: the following diagram

$$\begin{array}{ccc} \widehat{K}_m(X, \mu_n) & \xrightarrow{\Lambda_R \cdot \tau} & \widehat{K}_m(X_{\mu_n}, \mu_n)_\rho \otimes \mathbb{Q} \\ f_* \downarrow & & \downarrow f_{\mu_n *} \\ \widehat{K}_m(Y, \mu_n) & \longrightarrow & \widehat{K}_m(Y, \mu_n)_\rho \otimes \mathbb{Q} \end{array} \quad (5)$$

is commutative. In such a formulation, the equivariant R -genus again plays a crucial role.

To this aim, the definition of higher equivariant arithmetic K-groups and some reasonable technic that can be carried out for higher equivariant arithmetic K-theory should be clarified. We have settled these in [T3]. In fact, we have defined the higher equivariant arithmetic K-groups via the simplicial description of the Beilinson's regulators (cf. [BW]) and we have developed a localization sequence as well as an arithmetic concentration theorem. So, principally, we shall follow Thomason's approach to prove the commutativity of (5), but the fact that the direct image maps are only defined for the morphisms which are smooth over generic fibres will lead to a big gap comparing with the purely algebraic case. Some highly non-trivial analytic machinery should be involved, such as the transitivity property of analytic torsion forms and the Bismut-Ma's immersion formula.

The Köhler-Roessler's arithmetic Lefschetz fixed point formula has fruitful applications in number theory and in arithmetic geometry. One important reason is that the equivariant R -genus is closely related to the logarithmic derivative of certain L -functions. Köhler-Roessler and Maillot-Roessler have shown in [KR2] and in [MR1] that the Faltings heights and the periods of C.M. abelian varieties can be expressed as a formula in terms of the special value of logarithmic derivative of L -functions at 0. Further, in [MR2], Maillot-Roessler presented a series of conjectures about the relation between several invariants of arithmetic varieties and the special values of logarithmic derivative of Artin L -functions at negative integers. We hope that our Lefschetz-Riemann-Roch theorem for higher equivariant arithmetic K-groups would be helpful to understand these conjectures.

The structure of this article is as follows. In Section 2, we define the direct image maps between higher equivariant arithmetic K-groups. As an opportunity, we recall the analytic torsion for cubes of hermitian vector bundles introduced by D. Roessler in [Roe], actually our

construction is slightly different with but is equivalent to Roessler's construction. In Section 3, we discuss certain transitivity property of the direct image maps, the relation of equivariant analytic torsion forms with respect to families of submersions will be presented. In the last section, Section 4, we formulate and prove the commutativity of the diagram (5), an accurate computation via the deformation to the normal cone construction is given.

2 Higher equivariant arithmetic K-theory

2.1 Bott-Chern forms and arithmetic K-groups

Suggested by Soulé (cf. [So]), and also by Deligne (cf. [De]), the higher arithmetic K-groups of an arithmetic scheme X can be defined as the homotopy groups of the homotopy fibre of Beilinson's regulator map so that one obtains a long exact sequence

$$\cdots \longrightarrow \widehat{K}_m(X) \longrightarrow K_m(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} H_{\mathcal{D}}^{2p-m}(X, \mathbb{R}(p)) \longrightarrow \widehat{K}_{m-1}(X) \longrightarrow \cdots,$$

where $H_{\mathcal{D}}^*(X, \mathbb{R}(p))$ is the real Deligne-Beilinson cohomology and ch is the Beilinson's regulator map. In order to do this, a simplicial description of Beilinson's regulator map is necessary. In [BW], such a simplicial description was given by Burgos and Wang by using the higher Bott-Chern forms. Recently, in [T3], we followed Burgos-Wang's approach to define the higher equivariant Bott-Chern forms and further the higher equivariant arithmetic K-theory. In this subsection, we shall recall some relevant constructions and definitions, more details should be referred to [BW] and [T3].

At first, let X be a smooth algebraic variety over \mathbb{C} . We shall not distinguish X with its analytification $X(\mathbb{C})$, which is a complex manifold. Denote by $E_{\log}^*(X)$ the complex of differential forms on X with logarithmic singularities along infinity (cf. [T3, Definition 2.1]), then $E_{\log}^*(X)$ has a natural bigrading $E_{\log}^n(X) = \bigoplus_{p+q=n} E_{\log}^{p,q}(X)$ and this grading induces a Hodge filtration $F^p E_{\log}^n(X) = \bigoplus_{\substack{p' \geq p \\ p'+q'=n}} E_{\log}^{p',q'}(X)$. Write $E_{\log, \mathbb{R}}^*(X, p) := (2\pi i)^p E_{\log, \mathbb{R}}^*(X)$ with $E_{\log, \mathbb{R}}^*(X)$ the subcomplex of $E_{\log}^*(X)$ consisting of real forms, then we have a decomposition $E_{\log}^*(X) = E_{\log, \mathbb{R}}^*(X, p) \oplus E_{\log, \mathbb{R}}^*(X, p-1)$ and the projection $\pi_p : E_{\log}^*(X) \rightarrow E_{\log, \mathbb{R}}^*(X, p)$ is given by $\pi_p(x) = \frac{1}{2}(x + (-1)^p \bar{x})$. Moreover, for any $x \in E_{\log}^n(X)$, we define two filtered functions

$$F^{k,k}x = \sum_{l \geq k, l' \geq k} x^{l,l'} \quad \text{and} \quad F^k x = \sum_{l \geq k} x^{l,l'}.$$

Then we set $\pi(x) := \pi_{p-1}(F^{n-p+1, n-p+1}x)$.

The main result in [Bu1, Section 2] states that the following Dolbeault complex

$$\mathfrak{D}^n(E_{\log}(X), p) = \begin{cases} E_{\log, \mathbb{R}}^{n-1}(X, p-1) \cap \bigoplus_{\substack{p'+q'=n-1 \\ p' < p, q' < p}} E_{\log}^{p',q'}(X), & n < 2p; \\ E_{\log, \mathbb{R}}^n(X, p) \cap \bigoplus_{\substack{p'+q'=n \\ p' \geq p, q' \geq p}} E_{\log}^{p',q'}(X), & n \geq 2p, \end{cases}$$

with differential

$$d_{\mathcal{D}}x = \begin{cases} -\pi(dx), & n < 2p-1; \\ -2\partial\bar{\partial}x, & n = 2p-1; \\ dx, & n > 2p-1. \end{cases}$$

computes the real Deligne-Beilinson cohomology of X . Namely, one has

$$H_{\mathcal{D}}^n(X, \mathbb{R}(p)) = H^n(\mathfrak{D}^*(E_{\log}(X), p)).$$

We shall write $D^*(X, p) := \mathfrak{D}^*(E_{\log}(X), p)$ for short.

Moreover, let $x \in D^n(X, p)$ and $y \in D^m(X, q)$, we write $l = n + m$ and $r = p + q$. Then

$$x \bullet y = \begin{cases} (-1)^n r_p(x) \wedge y + x \wedge r_q(y), & n < 2p, m < 2q; \\ \pi(x \wedge y), & n < 2p, m \geq 2q, l < 2r; \\ F^{r,r}(r_p(x) \wedge y) + 2\pi_r \partial((x \wedge y)^{r-1, l-r}), & n < 2p, m \geq 2q, l \geq 2r; \\ x \wedge y, & n \geq 2p, m \geq 2q. \end{cases}$$

induces a product on $\bigoplus_p D^*(X, p)$ which is graded commutative and is associative up to chain homotopy. Here $r_p x = 2\pi_p(F^p dx)$ if $n \leq 2p-1$ and $r_p x = x$ otherwise. On the level of cohomology groups, this product coincides with the product defined by Beilinson. Notice that if $x \in D^{2p}(X, p)$ is a cocycle, then for all y, z we have $x \bullet y = y \bullet x$ and $y \bullet (x \bullet z) = (y \bullet x) \bullet z = x \bullet (y \bullet z)$.

In order to introduce the higher Bott-Chern form, let us construct a new complex $\tilde{D}^*(X, p)$ using the cocubical structure of the cartesian product of projective lines $(\mathbb{P}^1)^\cdot$. This complex $\tilde{D}^*(X, p)$ has the same cohomology groups as $D^*(X, p)$. Firstly one notices that $D^*(X \times (\mathbb{P}^1)^\cdot, p)$ form a cubical complex with face and degeneracy maps

$$d_i^j = (\text{Id} \times d_j^i)^* \quad \text{and} \quad s_i = (\text{Id} \times s^i)^*,$$

where

$$\begin{aligned} d_j^i : (\mathbb{P}^1)^k &\rightarrow (\mathbb{P}^1)^{k+1}, \quad i = 1, \dots, k, j = 0, 1, \\ s^i : (\mathbb{P}^1)^k &\rightarrow (\mathbb{P}^1)^{k-1}, \quad i = 1, \dots, k, \end{aligned}$$

which are given by

$$\begin{aligned} d_0^i(x_1, \dots, x_k) &= (x_1, \dots, x_{i-1}, (0:1), x_i, \dots, x_k), \\ d_1^i(x_1, \dots, x_k) &= (x_1, \dots, x_{i-1}, (1:0), x_i, \dots, x_k), \\ s^i(x_1, \dots, x_k) &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \end{aligned}$$

are the coface and the codegeneracy maps of $(\mathbb{P}^1)^\cdot$. Then we write $D_{\mathbb{P}}^{r,k}(X, p) = D^r(X \times (\mathbb{P}^1)^{-k}, p)$ and denote by $D_{\mathbb{P}}^{*,*}(X, p)$ the associated double complex with differentials

$$d' = d_{\mathfrak{D}} \quad \text{and} \quad d'' = \sum (-1)^{i+j-1} d_i^j.$$

Next, let $(x : y)$ be the homogeneous coordinates of \mathbb{P}^1 , and let $\omega = \partial\bar{\partial} \log \frac{x\bar{x} + y\bar{y}}{x\bar{x}} \in (2\pi i)E_{\mathbb{P}^1, \mathbb{R}}^2$ be a Kähler form over \mathbb{P}^1 . We shall write $\omega_i = p_i^* \omega \in E_{\log}^*(X \times (\mathbb{P}^1)^k)$ where $p_i : X \times (\mathbb{P}^1)^k \rightarrow \mathbb{P}^1, i = 1, \dots, k$ is the projection over the i -th projective line. The complex $\tilde{D}^*(X, p)$ is constructed by killing the degenerate classes and the classes coming from the projective spaces.

Definition 2.1. We define $\tilde{D}^*(X, p)$ as the associated simple complex of the double complex $\tilde{D}^{*,*}(X, p)$ which is given by

$$\tilde{D}^{r,k}(X, p) = D_{\mathbb{P}}^{r,k}(X, p) / \sum_{i=1}^{-k} s_i(D_{\mathbb{P}}^{r,k+1}(X, p)) \oplus \omega_i \wedge s_i(D_{\mathbb{P}}^{r-2,k+1}(X, p-1)).$$

The differential of this complex will be denoted by d .

A repetition of the proofs of [BW, Proposition 1.2 and Lemma 1.3] gives that the natural morphism of complexes

$$\iota : D^*(X, p) = \tilde{D}^{*,0}(X, p) \rightarrow \tilde{D}^*(X, p)$$

is a quasi-isomorphism.

Now, let X be a smooth μ_n -projective variety over \mathbb{C} and denote by $\mathcal{U} := \hat{\mathcal{P}}(X, \mu_n)$ the exact category of μ_n -equivariant vector bundles on X equipped with μ_n -invariant smooth hermitian metrics. We consider the exact cubes in the category \mathcal{U} . By definition, an exact k -cube in \mathcal{U} is a functor \mathcal{F} from $\langle -1, 0, 1 \rangle^n$, the k -th power of the ordered set $\langle -1, 0, 1 \rangle$, to \mathcal{U} such that for any $\alpha \in \langle -1, 0, 1 \rangle^{k-1}$ and $1 \leq i \leq k$, the 1-cube ∂_i^α defined by

$$\mathcal{F}_{\alpha_1, \dots, \alpha_{i-1}, -1, \alpha_i, \dots, \alpha_{k-1}} \rightarrow \mathcal{F}_{\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_i, \dots, \alpha_{k-1}} \rightarrow \mathcal{F}_{\alpha_1, \dots, \alpha_{i-1}, 1, \alpha_i, \dots, \alpha_{k-1}}$$

which is called an edge of \mathcal{F} is an short exact sequence. From now on, we shall write cubes instead of exact cubes for short. Let \mathcal{F} be a k -cube in \mathcal{U} , for $1 \leq i \leq k$ and $j \in \langle -1, 0, 1 \rangle$, the $(k-1)$ -cube $\partial_i^j \mathcal{F}$ defined by $(\partial_i^j \mathcal{F})_{\alpha_1, \dots, \alpha_{k-1}} = \mathcal{F}_{\alpha_1, \dots, \alpha_{i-1}, j, \alpha_i, \dots, \alpha_{k-1}}$ is called a face of \mathcal{F} . On the other hand, for any $1 \leq i \leq k+1$, we denote by $S_i^1 \mathcal{F}$ the $(k+1)$ -cube

$$(S_i^1 \mathcal{F})_{\alpha_1, \dots, \alpha_{k+1}} = \begin{cases} 0, & \alpha_i = 1; \\ \mathcal{F}_{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{k+1}}, & \alpha_i \neq 1, \end{cases}$$

such that the morphisms $(S_i^1 \mathcal{F})_{\alpha_1, \dots, \alpha_{i-1}, -1, \alpha_{i+1}, \dots, \alpha_{k+1}} \rightarrow (S_i^1 \mathcal{F})_{\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_{k+1}}$ are the identities of $(S_i^1 \mathcal{F})_{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{k+1}}$. Similarly, we have $(k+1)$ -cube $S_i^{-1} \mathcal{F}$.

Denote by $C_k \mathcal{U}$ the set of all k -cubes in \mathcal{U} , then we have the face maps $\partial_i^j : C_k \mathcal{U} \rightarrow C_{k-1} \mathcal{U}$ and the degeneracy maps $S_i^j : C_k \mathcal{U} \rightarrow C_{k+1} \mathcal{U}$. The cubes in the image of S_i^j are said to be degenerate. Let $\mathbb{Z}C_k \mathcal{U}$ be the free abelian group generated by $C_k \mathcal{U}$ and D_k be the subgroup of $\mathbb{Z}C_k \mathcal{U}$ generated by all degenerate k -cubes. Set $\tilde{\mathbb{Z}}C_k \mathcal{U} = \mathbb{Z}C_k \mathcal{U} / D_k$ and

$$d = \sum_{i=1}^k \sum_{j=-1}^1 (-1)^{i+j-1} \partial_i^j : \tilde{\mathbb{Z}}C_k \mathcal{U} \rightarrow \tilde{\mathbb{Z}}C_{k-1} \mathcal{U}.$$

Then $\tilde{\mathbb{Z}}C_* \mathcal{U} = (\tilde{\mathbb{Z}}C_* \mathcal{U}, d)$ is a homological complex.

Assume that \overline{E} is a hermitian k -cube in the category $\mathcal{U} = \hat{\mathcal{P}}(X, \mu_n)$. If \overline{E} is an emi-cube, namely the metrics on the quotient terms in all edges of \overline{E} are induced by the metrics on the middle terms (cf. [BW, Definition 3.5]), one can follow [BW, (3.7)] to associate a hermitian

locally free sheaf $\text{tr}_k(\overline{E})$ on $X \times (\mathbb{P}^1)^k$. This $\text{tr}_k(\overline{E})$ is called the k -transgression bundle of \overline{E} . If $k = 1$, as an emi-1-cube, \overline{E} is a short exact sequence

$$0 \longrightarrow \overline{E}_{-1} \xrightarrow{i} \overline{E}_0 \longrightarrow \overline{E}_1 \longrightarrow 0,$$

where the metric of \overline{E}_1 is induced by the metric of \overline{E}_0 . Then $\text{tr}_1(\overline{E})$ is the cokernel with quotient metric of the map $\overline{E}_{-1} \rightarrow \overline{E}_{-1} \otimes \mathcal{O}(1) \oplus \overline{E}_0 \otimes \mathcal{O}(1)$ by the rule $e_{-1} \mapsto e_{-1} \otimes \sigma_\infty \oplus i(e_{-1}) \otimes \sigma_0$. Here σ_0 (resp. σ_∞) is the canonical section of the tautological bundle $\mathcal{O}(1)$ on \mathbb{P}^1 which vanishes only at 0 (resp. ∞), and $\mathcal{O}(1)$ is endowed with the Fubini-Study metric. If $k > 1$, suppose that the transgression bundle is defined for $k - 1$. Let $\text{tr}_1(\overline{E})$ be the emi- $(k - 1)$ -cube over $X \times \mathbb{P}^1$ given by $\text{tr}_1(\overline{E})_\alpha = \text{tr}_1(\partial_1^\alpha(\overline{E}))$, then $\text{tr}_k(\overline{E})$ is defined as $\text{tr}_{k-1}(\text{tr}_1(\overline{E}))$.

Moreover, according to [BW, Proposition 3.6], for any hermitian cube \overline{E} in the category \mathcal{U} , there is a unique way to change the metrics on E_α for $\alpha \not\leq 0$ such that the obtained new hermitian cube is emi. In fact, for $i = 1, \dots, k$, define $\lambda_i^1 \overline{E}$ to be

$$(\lambda_i^1 \overline{E})_\alpha = \begin{cases} (E_\alpha, h_\alpha), & \text{if } \alpha_i = -1, 0; \\ (E_\alpha, h'_\alpha), & \text{if } \alpha_i = 1, \end{cases}$$

where h'_α is the metric induced by $h_{\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_k}$. Thus $\lambda_i^1 \overline{E}$ has the same locally free sheaves as \overline{E} , but the metrics on the face $\partial_i^1 \overline{E}$ are induced by the metrics of the face $\partial_i^0 \overline{E}$. To measure the difference between \overline{E} and $\lambda_i^1 \overline{E}$, let $\lambda_i^2(\overline{E})$ be the hermitian k -cube determined by $\partial_i^{-1} \lambda_i^2(\overline{E}) = \partial_i^1 \overline{E}$, $\partial_i^0 \lambda_i^2(\overline{E}) = \partial_i^1 \lambda_i^1(\overline{E})$, and $\partial_i^1 \lambda_i^2(\overline{E}) = 0$. Set $\lambda_i = \lambda_i^1 + \lambda_i^2$, $\lambda = \lambda_k \circ \dots \circ \lambda_1$ if $k \geq 1$ and $\lambda = \text{Id}$ otherwise. Then the map λ induces a morphism of complexes

$$\widetilde{\mathbb{Z}}C_*\mathcal{U} \rightarrow \widetilde{\mathbb{Z}}C_*^{\text{emi}}\mathcal{U}$$

which is the quasi-inverse of the inclusion $\widetilde{\mathbb{Z}}C_*^{\text{emi}}\mathcal{U} \hookrightarrow \widetilde{\mathbb{Z}}C_*\mathcal{U}$. To specify the μ_n -equivariant variety X , we shall write $\widetilde{\mathbb{Z}}C_*(X, \mu_n) := \widetilde{\mathbb{Z}}C_*\mathcal{U}$.

Definition 2.2. Define $R_n = \mathbb{R}$ if $n = 1$ and $R_n = \mathbb{C}$ otherwise. Fix a primitive n -th root of unity ζ_n , the equivariant higher Bott-Chern form associated to hermitian k -cube \overline{E} is defined as

$$\text{ch}_g^k(\overline{E}) := \text{ch}_g^0(\text{tr}_k(\lambda(\overline{E}))) \in \bigoplus_{p \geq 0} \widetilde{D}^*(X_{\mu_n}, p)[2p]_{R_n},$$

where $\text{ch}_g^0(F, h_F) = \sum_{l=1}^n \zeta_n^l \text{Tr}(\exp(-K_l))$ is the equivariant Chern-Weil form associated to an equivariant hermitian vector bundle $\overline{F}|_{X_{\mu_n}} = \bigoplus_{l=1}^n \overline{F}_l$ with curvature form K_l for \overline{F}_l . Moreover, replacing the equivariant Chern-Weil form by the equivariant Todd form $\text{Td}_g^0(F, h_F) = \text{Td}(F_n, h_{F_n}) \prod_{l \neq n} \det\left(\frac{1}{1 - \zeta_n^{-l} e^{K_l}}\right)$, we may define the equivariant higher Todd form as

$$\text{Td}_g^k(\overline{E}) := \text{Td}_g^0(\text{tr}_k(\lambda(\overline{E}))) \in \bigoplus_{p \geq 0} \widetilde{D}^*(X_{\mu_n}, p)[2p]_{R_n}.$$

Whence X is proper, Burgos and Wang gave in [BW, Section 6] a quasi-inverse $\varphi : \tilde{D}^*(X, p) \rightarrow D^*(X, p)$ of the quasi-isomorphism $\iota : D^*(X, p) \rightarrow \tilde{D}^*(X, p)$. By means of this quasi-inverse, the equivariant higher Bott-Chern form has another expression with value in $\bigoplus_{p \geq 0} D^*(X_{\mu_n}, p)[2p]_{R_n}$. To see this expression, let us set $z = x/y$ which defines the canonical coordinate map $\mathbb{C} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ by sending $z \rightarrow [z, 1]$. Then $\log |z|$ defines an L^1 function on $\mathbb{P}_{\mathbb{C}}^1$, which can be considered as a current. We shall denote by $\log |z_1|, \dots, \log |z_k|$ the corresponding currents on $(\mathbb{P}_{\mathbb{C}}^1)^k$. These currents can be formally considered as elements in $D^1((\mathbb{P}_{\mathbb{C}}^1)^k, 1)$, and they satisfy the following differential equation

$$d \log |z_j| = -2\partial\bar{\partial} \log |z_j| = -2i\pi(\delta_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \dots \times \{\infty\} \times \dots \times \mathbb{P}_{\mathbb{C}}^1} - \delta_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \dots \times \{0\} \times \dots \times \mathbb{P}_{\mathbb{C}}^1})$$

where ∞ and 0 stand at the j -th place. Let u_1, \dots, u_k be k elements in $\bigoplus_{p \geq 0} D^{2p-1}(\cdot, p)$, we define an element in $\bigoplus_{p \geq 0} D^{2p-k}(\cdot, p)$ by the formula

$$C_k(u_1, \dots, u_k) := -\left(\frac{1}{2}\right)^{k-1} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sigma} u_{\sigma(1)} \bullet (u_{\sigma(2)} \bullet (\dots u_{\sigma(k)})) \dots$$

where \mathfrak{S}_k stands for the k -th symmetric group. Then we have

$$dC_k(u_1, \dots, u_k) = \left(-\frac{1}{2}\right)k \sum_{j=1}^k (-1)^{j-1} d(u_j) \bullet C_{k-1}(u_1, \dots, \widehat{u}_j, \dots, u_k) \quad (6)$$

$$= \left(-\frac{1}{2}\right)k \sum_{j=1}^k (-1)^{j-1} d(u_j) \wedge C_{k-1}(u_1, \dots, \widehat{u}_j, \dots, u_k). \quad (7)$$

We refer to [Roe, Lemma 2.9] for a proof of these identities. With the above notations, the equivariant higher Bott-Chern form associated to hermitian k -cube \overline{E} with $k \geq 0$ is given by the expression

$$\varphi(\text{ch}_g^k(\overline{E})) = \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^k(\overline{E}) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2).$$

The same expression goes to the equivariant higher Todd form

$$\varphi(\text{Td}_g^k(\overline{E})) = \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{Td}_g^k(\overline{E}) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2).$$

Theorem 2.3. *The equivariant higher Bott-Chern forms induce a morphism of complexes*

$$\tilde{\mathbb{Z}}C^*(X, \mu_n) \xrightarrow{\lambda} \tilde{\mathbb{Z}}C_{\text{emi}}^*(X, \mu_n) \xrightarrow{\text{ch}_g^0 \text{otr}^*} \bigoplus_{p \geq 0} \tilde{D}^*(X_{\mu_n}, p)[2p]_{R_n} \xrightarrow{\varphi} \bigoplus_{p \geq 0} D^*(X_{\mu_n}, p)[2p]_{R_n},$$

which is denoted by ch_g . Here, $\tilde{\mathbb{Z}}C^*(X, \mu_n)$ and $\tilde{\mathbb{Z}}C_{\text{emi}}^*(X, \mu_n)$ are the (cohomological) complexes associated to the homological complexes $\tilde{\mathbb{Z}}C_*(X, \mu_n)$ and $\tilde{\mathbb{Z}}C_{\text{emi}*}^{\text{emi}}(X, \mu_n)$.

If X is a μ_n -equivariant arithmetic scheme over an arithmetic ring (D, Σ, F_∞) , we shall denote $X_{\mathbb{R}} := (X(\mathbb{C}), F_\infty)$ the real variety associated to X where F_∞ is the antiholomorphic involution of $X(\mathbb{C})$ induced by the conjugate-linear involution F_∞ over (D, Σ, F_∞) . For any sheaf of complex vector spaces V with a real structure over $X_{\mathbb{R}}$, we denote by σ the involution given by

$$\omega \mapsto \overline{F_\infty^*(\omega)}.$$

Write $D^*(X_{\mathbb{R}}, p) := D^*(X(\mathbb{C}), p)^\sigma$ for the subcomplex of $D^*(X(\mathbb{C}), p)$ consisting of the fixed elements under σ , we define the real Deligne-Beilinson cohomology of X as

$$H_D^*(X, \mathbb{R}(p)) := H^*(D^*(X_{\mathbb{R}}, p)).$$

Let us denote by $\widehat{\mathcal{P}}(X, \mu_n)$ the exact category of μ_n -equivariant hermitian vector bundles on X , and by $\widehat{S}(X, \mu_n)$ the simplicial set associated to the Waldhausen S -construction of $\widehat{\mathcal{P}}(X, \mu_n)$ (cf. [T3, Section 2.3]). The forgetful functor (forget about the metrics) $\pi : \widehat{\mathcal{P}}(X, \mu_n) \rightarrow \mathcal{P}(X, \mu_n)$ induces an equivalence of categories, so we have homotopy equivalence

$$|\widehat{S}(X, \mu_n)| \simeq |S(X, \mu_n)|$$

and isomorphisms of abelian groups

$$K_m(X, \mu_n) \cong \pi_{m+1}(|\widehat{S}(X, \mu_n)|, 0)$$

for any $m \geq 0$. To give the simplicial description of the equivariant regulator maps, we associate to each element in $S_k \widehat{\mathcal{P}}(X, \mu_n)$ a hermitian $(k-1)$ -cube. Firstly, notice that an element A in $S_k \widehat{\mathcal{P}}(X, \mu_n)$ is a family of injections

$$A_{0,1} \hookrightarrow A_{0,2} \hookrightarrow \cdots \hookrightarrow A_{0,k}$$

of μ_n -equivariant hermitian vector bundles on X with quotients $A_{i,j} \simeq A_{0,j}/A_{0,i}$ for each $i < j$. For $k=1$, we write

$$\text{Cub}(A_{0,1}) = A_{0,1}.$$

Suppose that the map Cub is defined for all $l < k$, then $\text{Cub}A$ is the $(k-1)$ -cube with

$$\begin{aligned} \partial_1^{-1} \text{Cub}A &= s_{k-2}^1 \cdots s_1^1(A_{0,1}), \\ \partial_1^1 \text{Cub}A &= \text{Cub}(\partial_0 A). \end{aligned}$$

Let $\mathbb{Z}\widehat{S}_*(X, \mu_n)$ be the simplicial abelian group generated by the simplicial set $\widehat{S}(X, \mu_n)$, and let $\mathcal{N}(\mathbb{Z}\widehat{S}_*(X, \mu_n))$ be the Moore complex associated to $\mathbb{Z}\widehat{S}_*(X, \mu_n)$ with differential $d = \sum_{i=0}^k (-1)^i \partial_i$ where ∂_i is the face map of $\widehat{S}(X, \mu_n)$. Then, according to [BW, Corollary 4.8], the map Cub defined above extends by linearity to a morphism of homological complexes

$$\text{Cub} : \mathcal{N}(\mathbb{Z}\widehat{S}_*(X, \mu_n)) \rightarrow \widetilde{\mathbb{Z}}C_*(X, \mu_n)[-1],$$

and hence one gets a simplicial map

$$\text{Cub} : \mathbb{Z}\widehat{S}_*(X, \mu_n) \rightarrow \mathcal{K}(\widetilde{\mathbb{Z}}C_*(X, \mu_n)[-1])$$

where \mathcal{K} is the Dold-Puppe functor.

Definition 2.4. Let notations and assumptions be as above. We denote by $D^{2p-*}(X_{\mu_n}, p)$ the homological complex associated to the complex $\tau_{\leq 0}(D^*(X_{\mu_n}, p)[2p])$ which is the canonical truncation of $D^*(X_{\mu_n}, p)[2p]$ at degree 0. We define a simplicial map

$$\begin{array}{ccc} \widetilde{\text{ch}}_g : \widehat{S}(X, \mu_n) & \xrightarrow{\text{Hu}} & \mathbb{Z}\widehat{S}_*(X, \mu_n) \\ & \downarrow \text{Cub} & \\ & \mathcal{K}(\widetilde{\mathbb{Z}C}_*(X, \mu_n)[-1]) & \xrightarrow{\mathcal{K}(\text{ch}_g)} \mathcal{K}(\bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)[-1]_{R_n}), \end{array}$$

where Hu is the Hurewicz map.

Definition 2.5. Let X be a μ_n -equivariant scheme over an arithmetic ring (D, Σ, F_∞) , and let X_{μ_n} be the fixed point subscheme. The higher equivariant arithmetic K-groups of X are defined as

$$\widehat{K}_m(X, \mu_n) := \pi_{m+1}(\text{homotopy fibre of } |\widetilde{\text{ch}}_g|) \quad \text{for } m \geq 1,$$

and the equivariant regulator maps

$$\text{ch}_g : K_m(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} H_D^{2p-m}(X_{\mu_n}, \mathbb{R}(p))_{R_n}$$

are defined as the homomorphisms induced by $\widetilde{\text{ch}}_g$ on the level of homotopy groups.

Remark 2.6. (i). We have the long exact sequence

$$\cdots \rightarrow \widehat{K}_m(X, \mu_n) \rightarrow K_m(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} H_D^{2p-m}(X_{\mu_n}, \mathbb{R}(p))_{R_n} \rightarrow \widehat{K}_{m-1}(X, \mu_n) \rightarrow \cdots$$

ending with

$$\begin{array}{ccc} \cdots \rightarrow K_1(X, \mu_n) & \longrightarrow & \bigoplus_{p \geq 0} H_D^{2p-1}(X_{\mu_n}, \mathbb{R}(p))_{R_n} \\ & & \downarrow \\ & & \pi_1(\text{homotopy fibre of } \widetilde{\text{ch}}_g) \longrightarrow K_0(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} H_D^{2p}(X_{\mu_n}, \mathbb{R}(p))_{R_n}. \end{array}$$

(ii). Whence $n = 1$, by the construction of the higher Bott-Chern forms, Definition 2.5 recovers the results given in [BW] for the non-equivariant case, namely

$$\text{ch}_g : K_m(X, \mu_1) \rightarrow \bigoplus_{p \geq 0} H_D^{2p-m}(X, \mathbb{R}(p))$$

equals the Beilinson's regulator map.

(iii). The higher equivariant arithmetic K-groups $\widehat{K}_m(X, \mu_n)$ can be defined for non-proper X , for details, see [T3, Section 2].

(iv). Let $s(\text{ch}_g)$ denote the simple complex associated to the chain morphism

$$\text{ch}_g : \quad \tilde{\mathbb{Z}}C_*(X, \mu_n) \xrightarrow{\text{ch}_g} \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n}.$$

Then, for any $m \geq 1$, there is an isomorphism

$$\hat{K}_m(X, \mu_n)_{\mathbb{Q}} \cong H_m(s(\text{ch}_g), \mathbb{Q}).$$

(v). A μ_n -equivariant hermitian sheaf on X is a μ_n -equivariant coherent sheaf on X which is locally free on $X(\mathbb{C})$ and is equipped with a μ_n -invariant hermitian metric. To a μ_n -equivariant hermitian sheaf, the higher equivariant Bott-Chern form can still be defined in the same way. Denote by $\hat{\mathcal{P}}'(X, \mu_n)$ the category of μ_n -equivariant hermitian sheaves on X , then instead of $\hat{\mathcal{P}}(X, \mu_n)$ one may define a new arithmetic K-theory $\hat{K}'_*(X, \mu_n)$ which is called the equivariant arithmetic K'-theory. Since $\hat{\mathcal{P}}'(X, \mu_n)$ and $\hat{\mathcal{P}}(X, \mu_n)$ define the same algebraic K-theory whence X is regular, it is easily seen from the Five-lemma that the natural inclusion $\hat{\mathcal{P}}(X, \mu_n) \subset \hat{\mathcal{P}}'(X, \mu_n)$ induces isomorphisms $\hat{K}_m(X, \mu_n) \cong \hat{K}'_m(X, \mu_n)$ for any $m \geq 1$.

2.2 Equivariant analytic torsion for hermitian cubes

In [BK], J.-M. Bismut and K. Köhler extended the Ray-Singer analytic torsion to the higher analytic torsion form T for a holomorphic submersion of complex manifolds. The purpose of making such an extension is that the analytic torsion form T satisfies a differential equation which gives a refinement of the Grothendieck-Riemann-Roch theorem on the level of smooth forms. Later, in [Mal], X. Ma generalized J.-M. Bismut and K. Köhler's results to the equivariant case. To the higher K-theory and the Deligne-Beilinson cohomology, a refinement of the Riemann-Roch theorem on the level of morphism of complexes representing the regulator maps needs an extension of higher analytic torsion for hermitian cubes, this is the content of [Roe]. In this subsection, we do the equivariant case by using Ma's equivariant analytic torsion forms. Our construction is slightly different with but is equivalent to Roessler's construction.

Let X, Y be two smooth μ_n -projective varieties over \mathbb{C} , and let $f : X \rightarrow Y$ be an equivariant, smooth and proper morphism. A Kähler fibration structure on f is a real closed $(1, 1)$ -form ω on X which induces Kähler metrics on the fibres of f (cf. [BK, Def. 1.1, Thm. 1.2]). For instance, we may fix a μ_n -invariant Kähler metric on X and choose corresponding Kähler form ω as a Kähler fibration structure on f . Let (E, h^E) be a μ_n -equivariant hermitian vector bundle on X such that E is f -acyclic i.e. the higher direct image $R^q f_* E$ vanishes for $q > 0$. The equivariant analytic torsion form $T_g(f, \omega, h^E)$ is an element of $\bigoplus_{p \geq 0} D^{2p-1}(Y_{\mu_n}, p)_{R_n}$, which depends on f, ω and (E, h^E) and satisfies the differential equation

$$dT_g(f, \omega, h^E) = \text{ch}_g(f_* E, f_* h^E) - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(Tf, h^{Tf}) \text{ch}_g(E, h^E)$$

where h^{Tf} is the hermitian metric induced by ω on the holomorphic tangent bundle Tf and r is the rank of the bundle Tf_{μ_n} . We would like to caution the reader that the equivariant analytic

torsion form we use here coincides with Ma's definition only up to a rescaling. In [Ma1, p. 1550], one defines an operator Φ which acts on differential forms. If we denote by $T'_g(f, \omega, h^E)$ Ma's equivariant torsion form, then the equality $2\Phi(T_g(f, \omega, h^E)) = T'_g(f, \omega, h^E)$ holds. From now on, we shall write $T_g(\omega, h^E)$ or $T_g(h^E)$ for $T_g(f, \omega, h^E)$, if there is no ambiguity about the underlying map or Kähler form. The following anomaly formula is useful for our later discussion.

Theorem 2.7. *Let ω' be the form associated to another Kähler fibration structure on $f : X \rightarrow Y$. Let h'^{Tf} be the metric on Tf induced by this new fibration. Then the following identity holds in $\bigoplus_{p \geq 0} (D^{2p-1}(Y_{\mu_n}, p)/\text{Im}d)$:*

$$\begin{aligned} T_g(\omega, h^E) - T_g(\omega', h^E) = & \text{ch}_g(f_*E, h^{f_*E}, h'^{f_*E}) \\ & - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(Tf, h^{Tf}, h'^{Tf}) \text{ch}_g(E, h^E) \end{aligned}$$

where $(f_*E, h^{f_*E}, h'^{f_*E})$ and (Tf, h^{Tf}, h'^{Tf}) stand for the emi-1-cubes of hermitian vector bundles

$$0 \longrightarrow (f_*E, h^{f_*E}) \xrightarrow{\text{Id}} (f_*E, h'^{f_*E}) \longrightarrow 0 \longrightarrow 0$$

and

$$0 \longrightarrow (Tf, h^{Tf}) \xrightarrow{\text{Id}} (Tf, h'^{Tf}) \longrightarrow 0 \longrightarrow 0.$$

Proof. This is [Ma1, Theorem 2.13]. □

According to the proof of [Ma1, Theorem 2.13], the differential form which measures the difference

$$T_g(\omega, h^E) - T_g(\omega', h^E) - \text{ch}_g(f_*E, h^{f_*E}, h'^{f_*E}) + \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(Tf, h^{Tf}, h'^{Tf}) \text{ch}_g(E, h^E)$$

in Theorem 2.7 can be explicitly written down. Let us denote by $\Delta(f, \overline{E}, \omega, \omega')$ this differential form, then it satisfies the differential equation

$$\begin{aligned} d\Delta(f, \overline{E}, \omega, \omega') = & T_g(\omega, h^E) - T_g(\omega', h^E) - \text{ch}_g(f_*E, h^{f_*E}, h'^{f_*E}) \\ & + \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(Tf, h^{Tf}, h'^{Tf}) \text{ch}_g(E, h^E). \end{aligned}$$

Now we consider the following setting. Let Z be a compact Kähler manifold and let Z_1 be a closed submanifold of Z . Choose a Kähler metric on Z and endow Z_1 with the restricted metric. Let $f_Z : X \times Z \rightarrow Y \times Z$ be the induced map and let ω, ω' be the Kähler forms of the product metrics on $X \times Z$ with respect to two Kähler fibrations on $f : X \rightarrow Y$. Similarly, let $f_{Z_1} : X \times Z_1 \rightarrow Y \times Z_1$ be the induced map and let ω_1, ω'_1 be the Kähler forms of the product metrics on $X \times Z_1$ with respect to the same two Kähler fibrations on $f : X \rightarrow Y$. We shall denote by j (resp. i) the natural embedding $X \times Z_1 \rightarrow X \times Z$ (resp. $Y \times Z_1 \rightarrow Y \times Z$). Then $j^*\omega = \omega_1$ and $j^*\omega' = \omega'_1$. Let \overline{E} be an f_Z -acyclic hermitian bundle on $X \times Z$, we have the following result.

Lemma 2.8. *The identity $i^* \Delta(f_Z, \overline{E}, \omega, \omega') = \Delta(f_{Z_1}, j^* \overline{E}, \omega_1, \omega'_1)$ holds.*

Proof. There are two ways to define the secondary characteristic forms $\text{ch}_g(f_* E, h^{f_* E}, h'^{f_* E})$ and $\text{Td}_g(Tf, h^{Tf}, h'^{Tf})$. The first one is using the 1-transgression bundle construction, and the second one is using the supertraces of Quillen's superconnections like the definition of analytic torsion, see [BGS]. The resulting forms are different but they induce the same class modulo $\text{Im} d$, and their difference is compatible with the restriction in the situation explained before this lemma, see the proof of the uniqueness of secondary characteristic class [BGS, §1. (f)]. On the other hand, using the second definition of the secondary characteristic forms i.e. the supertraces of Quillen's superconnections, the main contribution of the proof of [Ma1, Theorem 2.13] was to express the difference Δ as a limit of differential form which is naturally compatible with the restriction in the situation explained before this lemma. Therefore, we have the desired identity. \square

Roughly speaking, the equivariant analytic torsion for hermitian cubes is a chain homotopy of the following diagram of homological complexes

$$\begin{array}{ccc} \widetilde{\mathbb{Z}}C_*(X, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n} \\ \downarrow f_* & & \downarrow f_{\mu_n *} \circ \text{Td}_g(\overline{Tf}) \bullet (\cdot) \\ \widetilde{\mathbb{Z}}C_*(Y, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(Y_{\mu_n}, p)_{R_n}. \end{array} \quad (8)$$

Like the non-equivariant case treated in [Roe], the equivariant analytic torsion for hermitian cubes induces a commutative diagram on the level of homology groups and hence one gets an analytic proof of the equivariant version of Gillet's Riemann-Roch theorem for higher algebraic K-theory.

To construct a chain homotopy of (8), let us move in two steps. Notice that the equivariant higher Bott-Chern form factors as

$$\widetilde{\mathbb{Z}}C_*(X, \mu_n) \xrightarrow{\lambda} \widetilde{\mathbb{Z}}C_*^{\text{emi}}(X, \mu_n) \xrightarrow{\text{ch}_g^{\text{otr}*}} \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n},$$

we firstly clarify the difference between $f_*(\text{tr} \circ \lambda(\cdot))$ and $\text{tr} \circ \lambda(f_*(\cdot))$. Let \overline{E} be a hermitian k -cube in $\widehat{\mathcal{P}}(X, \mu_n)$. Since the Waldhausen K-theory space of $\widehat{\mathcal{P}}(X, \mu_n)$ is homotopy equivalent to the Waldhausen K-theory space of the full subcategory of $\widehat{\mathcal{P}}(X, \mu_n)$ consisting of f -acyclic bundles, we may assume that \overline{E} is f -acyclic. Then the hermitian bundles $f_*(\text{tr}_k \circ \lambda(\overline{E}))$ and $\text{tr}_k \circ \lambda(f_*(\overline{E}))$ are canonically isomorphic as bundles, but carry in general different metrics. In the following, we shall write $H(\overline{E})$ for the short exact sequence

$$0 \longrightarrow f_*(\text{tr}_k \circ \lambda(\overline{E})) \xrightarrow{\text{Id}} \text{tr}_k \circ \lambda(f_*(\overline{E})) \longrightarrow 0 \longrightarrow 0$$

which is an emi-1-cube of hermitian bundles on $Y \times (\mathbb{P}^1)^k$. The transgression bundle of $H(\overline{E})$ is a hermitian bundle on $Y \times (\mathbb{P}^1)^{k+1}$. But here we make a slight modification, let p_1 be the first

projection from $Y \times \mathbb{P}^1 \times (\mathbb{P}^1)^k$ to $Y \times (\mathbb{P}^1)^k$, we apply the transgression bundle construction to the short exact sequence $H(\overline{E})$ with respect to the projection p_1 to get a hermitian bundle on $Y \times (\mathbb{P}^1)^{k+1}$. With some abuse of notation, we still denote this hermitian bundle by $\text{tr}_1(H(\overline{E}))$ and it satisfies the following relations:

$$\text{tr}_1(H(\overline{E}))|_{Y \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_k \circ \lambda(f_*(\overline{E})), \quad \text{tr}_1(H(\overline{E}))|_{Y \times \{\infty\} \times (\mathbb{P}^1)^k} = f_*(\text{tr}_k \circ \lambda(\overline{E}))$$

and

$$\begin{aligned} \text{tr}_1(H(\overline{E}))|_{Y \times (\mathbb{P}^1)^i \times \{0\} \times (\mathbb{P}^1)^{k-i}} &= \text{tr}_1(H(\partial_i^0 \overline{E})), \\ \text{tr}_1(H(\overline{E}))|_{Y \times (\mathbb{P}^1)^i \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} &= \text{tr}_1(H(\partial_i^{-1} \overline{E})) \oplus \text{tr}_1(H(\partial_i^1 \overline{E})) \end{aligned}$$

for $i = 1, \dots, k$. Now we define

$$\Pi'_k(\overline{E}) := \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H(\overline{E}))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2).$$

The same reasoning as in [Roe, Lemma 3.3] proves that Π'_k vanishes on degenerate k -cubes, and hence we obtain a map $\Pi'_k : \widetilde{\mathbb{Z}}C_k(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} D^{2p-k-1}(Y_{\mu_n}, p)_{R_n}$ by linear extension.

Proposition 2.9. *The equality*

$$\begin{aligned} & d \circ \Pi'_k(\overline{E}) + \Pi'_{k-1} \circ d(\overline{E}) \\ &= \text{ch}_g(f_* \overline{E}) - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_*(\text{tr}_k \circ \lambda(\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \end{aligned}$$

holds.

Proof. We compute

$$\begin{aligned} d \circ \Pi'_k(\overline{E}) &= \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H(\overline{E}))) \wedge dC_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &= \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H(\overline{E}))) \wedge ((-\frac{1}{2})(k+1) \sum_{j=1}^{k+1} (-1)^{j-1} (-4\pi i) \\ &\quad (\delta_{z_j=\infty} - \delta_{z_j=0}) \wedge C_k(\log |z_1|^2, \dots, \log |z_j|^2, \dots, \log |z_{k+1}|^2)) \\ &= \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H(\overline{E}))) \wedge ((-\frac{1}{2})(k+1) \sum_{j=2}^{k+1} (-1)^{j-1} (-4\pi i) \\ &\quad (\delta_{z_j=\infty} - \delta_{z_j=0}) \wedge C_k(\log |z_1|^2, \dots, \log |z_j|^2, \dots, \log |z_{k+1}|^2)) \\ &\quad + \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(\text{tr}_k \circ \lambda(f_*(\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\ &\quad - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_*(\text{tr}_k \circ \lambda(\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^{k+1}}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left(\sum_{j=2}^{k+1} (-1)^{j-1} \text{ch}_g^0(\text{tr}_1(H(\partial_j^{-1}\overline{E} \oplus \partial_j^1\overline{E}))) - \text{ch}_g^0(\text{tr}_1(H(\partial_i^0\overline{E}))) \right) \wedge \\
&\quad C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\
&\quad + \text{ch}_g(f_*\overline{E}) \\
&\quad - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_*(\text{tr}_k \circ \lambda(\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\
&= \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(\text{tr}_1(H(-d\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\
&\quad + \text{ch}_g(f_*\overline{E}) \\
&\quad - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_*(\text{tr}_k \circ \lambda(\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\
&= -\Pi'_{k-1} \circ d(\overline{E}) + \text{ch}_g(f_*\overline{E}) \\
&\quad - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_*(\text{tr}_k \circ \lambda(\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2).
\end{aligned}$$

So we are done. \square

On the other hand, we equip $X \times (\mathbb{P}^1)^k$ with the product metric and we define

$$\Pi_k''(\overline{E}) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2)$$

where $T_g(\text{tr}_k \circ \lambda(\overline{E}))$ is the equivariant higher analytic torsion of the hermitian bundle $\text{tr}_k \circ \lambda(\overline{E})$ with respect to the fibration $f : X \times (\mathbb{P}^1)^k \rightarrow Y \times (\mathbb{P}^1)^k$. By [Roe, Lemma 3.5], the map Π_k'' vanishes on degenerate k -cubes and hence we obtain a map $\Pi_k'' : \widetilde{\mathbb{Z}}C_k(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} D^{2p-k-1}(Y_{\mu_n}, p)_{R_n}$ by linear extension.

Theorem 2.10. *Set $\Pi_k = \Pi'_k + \Pi_k''$, then Π_k decides a chain homotopy of the diagram (8). This map $\Pi_k : \widetilde{\mathbb{Z}}C_k(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} D^{2p-k-1}(Y_{\mu_n}, p)_{R_n}$ is called the equivariant higher analytic torsion for hermitian cubes.*

Proof. Let \overline{E} be a hermitian k -cube in $\widetilde{\mathbb{Z}}C_k(X, \mu_n)$, we compute

$$\begin{aligned}
&d \circ \Pi_k(\overline{E}) + \Pi_{k-1} \circ d(\overline{E}) \\
&= d \circ \Pi'_k(\overline{E}) + \Pi'_{k-1} \circ d(\overline{E}) + d \circ \Pi_k''(\overline{E}) + \Pi_{k-1}'' \circ d(\overline{E}) \\
&= \text{ch}_g(f_*\overline{E}) - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_*(\text{tr}_k \circ \lambda(\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\
&\quad + d \circ \Pi_k''(\overline{E}) + \Pi_{k-1}'' \circ d(\overline{E}).
\end{aligned}$$

and

$$d \circ \Pi_k''(\overline{E}) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} dC_{k+1}(T_g(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2)$$

$$\begin{aligned}
&= \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left(-\frac{1}{2}\right)(k+1) \left(\sum_{j=1}^k (-1)^j (-4\pi i)\right. \\
&\quad \left.(\delta_{z_j=\infty} - \delta_{z_j=0}) \wedge C_k(T_g(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_j|^2, \dots, \log |z_k|^2)\right. \\
&\quad \left.+ dT_g(\text{tr}_k \circ \lambda(\overline{E})) \bullet C_k(\log |z_1|^2, \dots, \log |z_k|^2)\right) \\
&= \frac{(-1)^k}{k!(2\pi i)^{k-1}} \int_{(\mathbb{P}^1)^{k-1}} \sum_{j=1}^k (-1)^j [C_k(T_g(\text{tr}_{k-1} \circ \lambda(\partial_j^0 \overline{E})), \log |z_1|^2, \dots, \log |z_{k-1}|^2) \\
&\quad - C_k(T_g(\text{tr}_{k-1} \circ \lambda(\partial_j^{-1} \overline{E}) \oplus \text{tr}_{k-1} \circ \lambda(\partial_j^1 \overline{E})), \log |z_1|^2, \dots, \log |z_{k-1}|^2)] \\
&\quad + \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} [\text{ch}_g^0(f_*(\text{tr}_k \circ \lambda(\overline{E}))) - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Y_{\mu_n} \times (\mathbb{P}^1)^k} \text{Td}_g(\overline{Tf}) \\
&\quad \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E}))] \bullet C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\
&= -\Pi''_{k-1} \circ d(\overline{E}) \\
&\quad + \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_*(\text{tr}_k \circ \lambda(\overline{E}))) \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\
&\quad - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(\overline{Tf}) \bullet \text{ch}_g(\overline{E}).
\end{aligned}$$

Combining these two computations, we finally get

$$d \circ \Pi_k(\overline{E}) + \Pi_{k-1} \circ d(\overline{E}) = \text{ch}_g(f_* \overline{E}) - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(\overline{Tf}) \bullet \text{ch}_g(\overline{E}).$$

So we are done. \square

If we are given another fibration structure ω' , then for any f -acyclic hermitian k -cube \overline{E} in $\widehat{\mathcal{P}}(X, \mu_n)$, the short exact sequence

$$0 \longrightarrow (f_* E, h'^{f_* E}) \xrightarrow{\text{Id}} (f_* E, h^{f_* E}) \longrightarrow 0 \longrightarrow 0$$

forms a hermitian $(k+1)$ -cube $H_f(\overline{E})$ on Y such that the transgression bundle $\text{tr}_{k+1}(\lambda(H_f(\overline{E})))$ satisfies the relation

$$\begin{aligned}
&\text{tr}_{k+1}(\lambda(H_f(\overline{E})))|_{Y \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_k(\lambda(f_* E, h^{f_* E})), \\
&\text{tr}_{k+1}(\lambda(H_f(\overline{E})))|_{Y \times \{\infty\} \times (\mathbb{P}^1)^k} = \text{tr}_k(\lambda(f_* E, h'^{f_* E}))
\end{aligned}$$

and

$$\begin{aligned}
&\text{tr}_{k+1}(\lambda(H_f(\overline{E})))|_{Y \times (\mathbb{P}^1)^i \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_k(\lambda(H_f(\partial_i^0 \overline{E}))), \\
&\text{tr}_{k+1}(\lambda(H_f(\overline{E})))|_{Y \times (\mathbb{P}^1)^i \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_k(\lambda(H_f(\partial_i^{-1} \overline{E}))) \oplus \text{tr}_k(\lambda(H_f(\partial_i^1 \overline{E})))
\end{aligned}$$

for $i = 1, \dots, k$. Therefore, the following map

$$\Pi_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(H_f(\overline{E}))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2)$$

which vanishes on degenerate cubes provides a chain homotopy of homological complexes between the maps $\text{ch}_g \circ f_*$ and $\text{ch}_g \circ f'_*$ where $f'_*(\overline{E}) := (f_*E, h'^{f_*E})$ is the push-forward with respect to the new fibration ω' . Similarly, by projection formula, the map

$$\Pi_k^{(3)}(\overline{E}) := \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left[\frac{1}{(2\pi i)^r} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Y_{\mu_n} \times (\mathbb{P}^1)^k} \text{Td}_g(Tf, h^{Tf}, h'^{Tf}) \right. \\ \left. \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E})) \right] \bullet C_k(\log |z_1|^2, \dots, \log |z_k|^2)$$

gives a chain homotopy of homological complexes between the maps $f_{\mu_n*} \circ (\text{Td}_g(Tf, h^{Tf}) \bullet \text{ch}_g)$ and $f_{\mu_n*} \circ (\text{Td}_g(Tf, h'^{Tf}) \bullet \text{ch}_g)$. Finally we write $\Pi_k^{(2)} = \Pi_k'^{(2)} + \Pi_k''^{(2)}$ for the chain homotopy defined in Theorem 2.10 between the maps $\text{ch}_g \circ f'_*$ and $f_{\mu_n*} \circ (\text{Td}_g(Tf, h'^{Tf}) \bullet \text{ch}_g)$ with respect to the new fibration ω' . Then $\Pi_k^{(1)} + \Pi_k^{(2)} - \Pi_k^{(3)}$ decides a chain homotopy between $\text{ch}_g \circ f_*$ and $f_{\mu_n*} \circ (\text{Td}_g(Tf, h^{Tf}) \bullet \text{ch}_g)$. At the last of this subsection, we compare this homotopy $\Pi_k^{(1)} + \Pi_k^{(2)} - \Pi_k^{(3)}$ with Π_k constructed in Theorem 2.10.

Definition 2.11. Let f, g be two morphisms of homological complexes $A_* \rightarrow B_*$, and let h_1, h_2 be two chain homotopies between f and g . We say that h_1 is homotopic to h_2 if there exists a map $H : A_* \rightarrow B_{*+2}$ satisfying the condition that $Hd - dH = h_1 - h_2$.

Now, we denote by $H_f^{f'}(\overline{E})$ the following emi-2-cube of hermitian bundles on $Y \times (\mathbb{P}^1)^k$

$$\begin{array}{ccccc} f'_*(\text{tr}_k \circ \lambda(\overline{E})) & \xrightarrow{\text{Id}} & \text{tr}_k \circ \lambda(f'_*(\overline{E})) & \longrightarrow & 0 \\ \downarrow \text{Id} & & \downarrow \text{Id} & & \downarrow \\ f_*(\text{tr}_k \circ \lambda(\overline{E})) & \xrightarrow{\text{Id}} & \text{tr}_k \circ \lambda(f_*(\overline{E})) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Then we construct a hermitian bundle $\text{tr}_2(H_f^{f'}(\overline{E}))$ on $Y \times (\mathbb{P}^1)^{k+2}$ as a modified second transgression bundle of $H_f^{f'}(\overline{E})$ such that it satisfies the following relations:

$$\text{tr}_2(H_f^{f'}(\overline{E}))|_{Y \times \{0\} \times (\mathbb{P}^1)^{k+1}} = \text{tr}_{k+1}(\lambda(H_f(\overline{E}))),$$

$$\text{tr}_2(H_f^{f'}(\overline{E}))|_{Y \times \{\infty\} \times (\mathbb{P}^1)^{k+1}} = \text{tr}_1(H_f(\text{tr}_k \circ \lambda(\overline{E}))),$$

$$\text{tr}_2(H_f^{f'}(\overline{E}))|_{Y \times \mathbb{P}^1 \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_1(H(\overline{E})), \quad \text{tr}_2(H_f^{f'}(\overline{E}))|_{Y \times \mathbb{P}^1 \times \{\infty\} \times (\mathbb{P}^1)^k} = \text{tr}_1(H'(\overline{E}))$$

and

$$\text{tr}_2(H_f^{f'}(\overline{E}))|_{Y \times (\mathbb{P}^1)^{i+1} \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_2(H_f^{f'}(\partial_i^0 \overline{E})),$$

$$\text{tr}_2(H_f^{f'}(\overline{E}))|_{Y \times (\mathbb{P}^1)^{i+1} \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_2(H_f^{f'}(\partial_i^{-1} \overline{E})) \oplus \text{tr}_2(H_f^{f'}(\partial_i^1 \overline{E}))$$

for $i = 1, \dots, k$. We set

$$\Pi_{f,k}^{f'}(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}_g^0(\text{tr}_2(H_f^{f'}(\overline{E}))) \wedge C_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2).$$

Then $\Pi_{f,k}^{f'}$ vanishes on degenerate k -cubes, and we obtain a map

$$\Pi_{f,k}^{f'} : \widetilde{\mathbb{Z}}C_k(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} D^{2p-k-2}(Y_{\mu_n}, p)_{R_n}$$

by linear extension.

Proposition 2.12. *Let notations and assumptions be as above. Then the chain homotopy Π_k is homotopic to the chain homotopy $\Pi_k^{(1)} + \Pi_k^{(2)} - \Pi_k^{(3)}$.*

Proof. Firstly, we set

$$\begin{aligned} \Pi_k^{(3')}(\overline{E}) := & \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\frac{1}{(2\pi i)^r} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Y_{\mu_n} \times (\mathbb{P}^1)^k} \text{Td}_g(Tf, h^{Tf}, h'^{Tf}) \right. \\ & \left. \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2 \right). \end{aligned}$$

It also decides a chain homotopy between the maps $f_{\mu_n*} \circ (\text{Td}_g(Tf, h^{Tf}) \bullet \text{ch}_g)$ and $f_{\mu_n*} \circ (\text{Td}_g(Tf, h'^{Tf}) \bullet \text{ch}_g)$. Since the product \bullet on Deligne complex is graded commutative and is associative up to natural homotopy, we claim that $\Pi_k^{(3')}(\overline{E})$ is naturally homotopic to $\Pi_k^{(3)}(\overline{E})$ so that we are left to show that Π_k is homotopic to $\Pi_k^{(1)} + \Pi_k^{(2)} - \Pi_k^{(3')}$. Actually, our claim follows from the fact that $d\Pi_k^{(3)}(\overline{E}) - d\Pi_k^{(3')}(\overline{E}) = \Pi_k^{(3)}(-d\overline{E}) - \Pi_k^{(3')}(-d\overline{E})$ and [T3, Remark 2.4, Lemma 2.5].

Now, let \overline{E} be a hermitian k -cube in $\widehat{\mathcal{P}}(X, \mu_n)$ which is f -acyclic. We compute

$$\begin{aligned} d \circ \Pi_{f,k}^{f'}(\overline{E}) &= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}_g^0(\text{tr}_2(H_f^{f'}(\overline{E}))) \wedge dC_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2) \\ &= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}_g^0(\text{tr}_2(H_f^{f'}(\overline{E}))) \wedge [(-\frac{1}{2})(k+2) \sum_{j=1}^{k+2} (-1)^{j-1} (-4\pi i) \\ &\quad (\delta_{z_j=\infty} - \delta_{z_j=0}) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |\widehat{z_j}|^2, \dots, \log |z_{k+2}|^2)] \\ &= \Pi_{f,k-1}^{f'} \circ d(\overline{E}) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} [\text{ch}_g^0(\text{tr}_1(H(\overline{E}))) - \text{ch}_g^0(\text{tr}_1(H'(\overline{E})))] \wedge \\ &\quad C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &\quad + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} [\text{ch}_g^0(\text{tr}_{k+1}(\lambda(H_f(\overline{E})))) - \text{ch}_g^0(\text{tr}_1(H_f(\text{tr}_k \circ \lambda(\overline{E}))))] \wedge \\ &\quad C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \end{aligned}$$

$$\begin{aligned}
&= \Pi_{f,k-1}^{f'} \circ d(\overline{E}) - \Pi_k'(\overline{E}) + \Pi_k^{(2)}(\overline{E}) \\
&\quad + \Pi_k^{(1)}(\overline{E}) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H_f(\text{tr}_k \circ \lambda(\overline{E})))) \wedge \\
&\quad C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2).
\end{aligned}$$

On the other hand, according to the anomaly formula Theorem 2.7, we have

$$\begin{aligned}
&\Pi_k''(\overline{E}) - \Pi_k^{(2)}(\overline{E}) \\
&= \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2) \\
&\quad - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g'(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2) \\
&= \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\frac{1}{4\pi i} \int_{Y_{\mu_n} \times (\mathbb{P}^1)^{k+1} / Y_{\mu_n} \times (\mathbb{P}^1)^k} \text{ch}_g^0(\text{tr}_1(H_f(\text{tr}_k \circ \lambda(\overline{E})))) \log |z_0|^2, \right. \\
&\quad \log |z_1|^2, \dots, \log |z_k|^2 \Big) \\
&\quad - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\frac{1}{(2\pi i)^r} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Y_{\mu_n} \times (\mathbb{P}^1)^k} \text{Td}_g(Tf, h^{Tf}, h'^{Tf}) \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E})), \right. \\
&\quad \log |z_1|^2, \dots, \log |z_k|^2 \Big) \\
&\quad + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d\Delta(f, \text{tr}_k \circ \lambda(\overline{E}), \omega, \omega'), \log |z_1|^2, \dots, \log |z_k|^2) \\
&= \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H_f(\text{tr}_k \circ \lambda(\overline{E})))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
&\quad + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d\Delta(f, \text{tr}_k \circ \lambda(\overline{E}), \omega, \omega'), \log |z_1|^2, \dots, \log |z_k|^2) \\
&\quad - \Pi_k^{(3')}(\overline{E}).
\end{aligned}$$

We formally define a product $C_{k+1}(\Delta(f, \text{tr}_k \circ \lambda(\overline{E}), \omega, \omega'), \log |z_1|^2, \dots, \log |z_k|^2)$ in a similar way to $C_{k+1}(\cdot, \dots, \cdot)$ like follows.

$$\begin{aligned}
&C_{k+1}(\Delta(f, \text{tr}_k \circ \lambda(\overline{E}), \omega, \omega'), \log |z_1|^2, \dots, \log |z_k|^2) \tag{9} \\
&= - \left(-\frac{1}{2}\right)^k \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \Delta \bullet (\log |z_{\sigma(1)}|^2 \bullet (\log |z_{\sigma(2)}|^2 \bullet (\dots \log |z_{\sigma(k)}|^2) \dots)) \\
&\quad - \left(-\frac{1}{2}\right)^k \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \log |z_{\sigma(1)}|^2 \bullet (\Delta \bullet (\log |z_{\sigma(2)}|^2 \bullet (\dots \log |z_{\sigma(k)}|^2) \dots)) \\
&\quad \dots \\
&\quad - \left(-\frac{1}{2}\right)^k \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \log |z_{\sigma(1)}|^2 \bullet (\log |z_{\sigma(2)}|^2 \bullet (\dots \log |z_{\sigma(k)}|^2 \bullet \Delta) \dots)
\end{aligned}$$

Then we set

$$\Delta_k(\overline{E}) = \frac{(-1)^k}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(\Delta(f, \text{tr}_k \circ \lambda(\overline{E}), \omega, \omega'), \log |z_1|^2, \dots, \log |z_k|^2),$$

and it is readily checked by Lemma 2.8 that

$$\Delta_{k-1}(d\overline{E}) - d\Delta_k(\overline{E}) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d\Delta(f, \text{tr}_k \circ \lambda(\overline{E}), \omega, \omega'), \log |z_1|^2, \dots, \log |z_k|^2).$$

Combing all the above computations, we finally get

$$\begin{aligned} & (\Pi_{f,k-1}^{f'} + \Delta_{k-1}) \circ d(\overline{E}) - d \circ (\Pi_{f,k}^{f'} + \Delta_k)(\overline{E}) \\ &= -\Pi_k^{(2)}(\overline{E}) + \Pi_k'(\overline{E}) - \Pi_k^{(1)}(\overline{E}) + \Delta_{k-1}(d\overline{E}) - d\Delta_k(\overline{E}) \\ & \quad + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H_f(\text{tr}_k \circ \lambda(\overline{E})))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &= -\Pi_k^{(2)}(\overline{E}) + \Pi_k'(\overline{E}) - \Pi_k^{(1)}(\overline{E}) - \Pi_k^{(2)}(\overline{E}) + \Pi_k''(\overline{E}) + \Pi_k^{(3')}(\overline{E}) \\ &= \Pi_k(\overline{E}) - (\Pi_k^{(1)}(\overline{E}) + \Pi_k^{(2)}(\overline{E}) - \Pi_k^{(3')}(\overline{E})). \end{aligned}$$

So we are done. \square

2.3 Direct image map between arithmetic K-groups

In this subsection, we define the direct image map between arithmetic K-groups of μ_n -equivariant arithmetic schemes by means of the equivariant higher analytic torsion for hermitian cubes constructed in last subsection.

Let now X and Y be two μ_n -equivariant schemes over an arithmetic ring (D, Σ, F_∞) . Assume that $f : X \rightarrow Y$ is an equivariant, proper and flat morphism from X to Y such that f is smooth over generic fibre. Notice that the chain homotopy

$$\Pi_* : \widetilde{\mathbb{Z}}C_*(X(\mathbb{C}), \mu_n) \rightarrow \bigoplus_{p \geq 0} D^{2p-* - 1}(Y(\mathbb{C})_{\mu_n}, p)_{R_n}$$

is σ -invariant and the following diagrams

$$\begin{array}{ccccc} \widehat{S}^{f\text{-acyclic}}(X, \mu_n) & \xrightarrow{\text{Hu}} & \mathbb{Z}\widehat{S}_*(X, \mu_n) & \xrightarrow{\text{Cub}} & \mathcal{K}(\widetilde{\mathbb{Z}}C_*(X, \mu_n)[-1]) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ \widehat{S}(Y, \mu_n) & \xrightarrow{\text{Hu}} & \mathbb{Z}\widehat{S}_*(Y, \mu_n) & \xrightarrow{\text{Cub}} & \mathcal{K}(\widetilde{\mathbb{Z}}C_*(Y, \mu_n)[-1]) \end{array}$$

are naturally commutative, we obtain a square of simplicial sets

$$\begin{array}{ccc} \widehat{S}(X, \mu_n) & \xrightarrow{\widetilde{\text{ch}}_g} & \mathcal{K}(\bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)[-1]_{R_n}) \\ \downarrow f_* & & \downarrow f_{\mu_n*} \circ \text{Td}_g(\overline{Tf}) \bullet (\cdot) \\ \widehat{S}(Y, \mu_n) & \xrightarrow{\widetilde{\text{ch}}_g} & \mathcal{K}(\bigoplus_{p \geq 0} D^{2p-*}(Y_{\mu_n}, p)[-1]_{R_n}), \end{array}$$

which is commutative up to an explicit homotopy. Applying the geometric realization construction to the above square, we get a continuous map between homotopy fibres

$$|f| : \text{homotopy fibre of } |\widetilde{\text{ch}}_g^X| \longrightarrow \text{homotopy fibre of } |\widetilde{\text{ch}}_g^Y|.$$

Definition 2.13. For $m \geq 1$, the direct image map $f_* : \widehat{K}_m(X, \mu_n) \rightarrow \widehat{K}_m(Y, \mu_n)$ is defined as the homomorphism of abelian groups deduced by the map $|f|$ on the level of homotopy groups.

Remark 2.14. The condition “flatness” of the map f is only used to guarantee that the direct image of an f -acyclic bundle is locally free. By introducing the arithmetic K' -theory and using the isomorphisms $\widehat{K}_m(X, \mu_n) \cong \widehat{K}'_m(X, \mu_n)$ which hold for regular schemes, the condition “flatness” can certainly be removed.

To study the direct image map up to torsion, we need the following lemma.

Lemma 2.15. Consider the following diagram of homological complexes

$$\begin{array}{ccc} A_* & \xrightarrow{i} & B_* \\ f_1 \downarrow & & \downarrow g_1 \\ C_* & \xrightarrow{j} & D_* \end{array} \quad \begin{array}{c} f_2 \\ g_2 \end{array}$$

Assume that $j \circ f_1$ (resp. $j \circ f_2$) is homotopic to $g_1 \circ i$ (resp. $g_2 \circ i$) via the chain homotopy h_1 (resp. h_2), and that f_1 (resp. g_1) is homotopic to f_2 (resp. g_2) via the chain homotopy π_f (resp. π_g). Suppose that the chain homotopy $j \circ \pi_f + h_2 - \pi_g \circ i$ is homotopic to the chain homotopy h_1 , then the morphism on simple complexes

$$(f_1, g_1, h_1) : s_*(i : A_* \rightarrow B_*) \rightarrow s_*(j : C_* \rightarrow D_*)$$

is chain homotopic to (f_2, g_2, h_2) .

Proof. Let $(a, b) \in A_k \oplus B_{k+1}$, the morphism (f_1, g_1, h_1) (resp. (f_2, g_2, h_2)) sends (a, b) to $(f_1(a), g_1(b) + h_1(a))$ (resp. $(f_2(a), g_2(b) + h_2(a))$). Let $H : A_* \rightarrow D_{*+2}$ be the homotopy such that

$$Hd - dH = h_1 - (j \circ \pi_f + h_2 - \pi_g \circ i),$$

and we define $\widetilde{H}(a, b) = (\pi_f(a), -\pi_g(b) + H(a))$. Then we compute

$$d\widetilde{H}(a, b) = d(\pi_f(a), -\pi_g(b) + H(a))$$

$$\begin{aligned}
&= (d\pi_f(a), j \circ \pi_f(a) + d\pi_g(b) - dH(a)) \\
&= (f_1(a) - f_2(a) - \pi_f(da), g_1(b) - g_2(b) - \pi_g(db) - Hd(a) + h_1(a) - h_2(a) + \pi_g \circ i(a)) \\
&= (f_1(a), g_1(b) + h_1(a)) - (f_2(a), g_2(b) + h_2(a)) - (\pi_f da, \pi_g db) - \pi_g \circ i(a) + Hd(a) \\
&= (f_1(a), g_1(b) + h_1(a)) - (f_2(a), g_2(b) + h_2(a)) - \tilde{H}(da, i(a) - db) \\
&= (f_1(a), g_1(b) + h_1(a)) - (f_2(a), g_2(b) + h_2(a)) - \tilde{H}d(a, b).
\end{aligned}$$

So we are done. \square

Corollary 2.16. *Let notations and assumptions be as above, then the direct image map $f_* : \widehat{K}_m(X, \mu_n)_{\mathbb{Q}} \rightarrow \widehat{K}_m(Y, \mu_n)_{\mathbb{Q}}$ without torsion is independent of the choice of the Kähler fibration structure.*

Proof. This follows from Remark 2.6 (iv), Theorem 2.12 and Lemma 2.15. \square

3 Transitivity of the direct image maps

Let $f : X \rightarrow Y$, $h : Y \rightarrow Z$ and $g : X \rightarrow Z$ be three equivariant and proper morphisms between μ_n -equivariant schemes, which are all smooth over the generic fibres. Assume that $g = h \circ f$, in this section, we shall compare the direct image map g_* with the composition $h_* \circ f_*$. To this aim, we shall firstly discuss the functoriality of the equivariant analytic torsion forms with respect to a composition of submersions.

3.1 Analytic torsion forms and families of submersions

Let W, V and S be three μ_n -equivariant smooth algebraic varieties over \mathbb{C} with $S = S_{\mu_n}$. Suppose that $f : W \rightarrow V$ and $h : V \rightarrow S$ are two proper smooth morphisms, then passing to their analytifications the maps $f : W(\mathbb{C}) \rightarrow V(\mathbb{C})$ and $h : V(\mathbb{C}) \rightarrow S(\mathbb{C})$ are holomorphic submersions with compact fibres. Set $g = h \circ f$, it is also a proper smooth morphism and $g : W(\mathbb{C}) \rightarrow S(\mathbb{C})$ is a holomorphic submersion with compact fibre as well.

Let ω^W and ω^V be two μ_n -invariant Kähler forms on W and on V . As before, ω^W and ω^V decide Kähler fibration structures on the morphisms f, h and g and they induce μ_n -invariant hermitian metrics on relative tangent bundles Tf, Th and Tg . Consider the following short exact sequence of hermitian vector bundles

$$\overline{T}(f, h, h \circ f) : 0 \rightarrow \overline{T}f \rightarrow \overline{T}g \rightarrow f^* \overline{T}h \rightarrow 0,$$

it can be regarded as an 1-cube of hermitian bundles on W . Then the equivariant higher Todd form $\text{Td}_g(\overline{T}(f, h, h \circ f))$ has been defined and it satisfies the differential equation

$$d\text{Td}_g(\overline{T}(f, h, h \circ f)) = \text{Td}_g(\overline{T}g) - f_{\mu_n}^* \text{Td}_g(\overline{T}h) \text{Td}_g(\overline{T}f).$$

Now let \overline{E} be a hermitian vector bundle on W , we shall assume that E is f -acyclic and g -acyclic. Then the Leray spectral sequence $E_2^{i,j} = R^i h_*(R^j f_* E)$ degenerates from E_2 so that

$f_*E = R^0 f_*(E)$ is h -acyclic and $g_*E \cong h_*f_*E$. Clearly, g_*E and h_*f_*E carry in general different L^2 metrics. Consider the following short exact sequence of hermitian vector bundles

$$\overline{E}(f, h, h \circ f) : 0 \rightarrow h_*f_*\overline{E} \rightarrow g_*\overline{E} \rightarrow 0 \rightarrow 0,$$

it can be regarded as an emi-1-cube of hermitian bundles on S . Then the equivariant higher Bott-Chern form $\text{ch}_g(\overline{E}(f, h, h \circ f))$ satisfies the differential equation

$$d\text{ch}_g(\overline{E}(f, h, h \circ f)) = \text{ch}_g(g_*\overline{E}) - \text{ch}_g(h_*f_*\overline{E}).$$

The main result in this subsection is the following.

Theorem 3.1. *Let notations and assumptions be as above. Then the following identity holds in $\bigoplus_{p \geq 0} (D^{2p-1}(S, p)/\text{Im}d)$:*

$$\begin{aligned} T_g(g, \omega^W, h^E) - T_g(h, \omega^V, h^{f_*E}) - \frac{1}{(2\pi i)^{r_h}} \int_{V_{\mu_n}/S} \text{Td}_g(\overline{Th}) T_g(f, \omega^W, h^E) \\ = \text{ch}_g(\overline{E}(f, h, h \circ f)) - \frac{1}{(2\pi i)^{r_g}} \int_{W_{\mu_n}/S} \text{Td}_g(\overline{T}(f, h, h \circ f)) \text{ch}_g(\overline{E}) \end{aligned}$$

where r_h and r_g are the relative dimensions of V_{μ_n}/S and W_{μ_n}/S respectively.

Proof. This is a natural extension of [Ma2, Théorème 3.5] to the equivariant case. To prove this extension, one may follow the same approach as Ma in [Ma2, Section 4-Section 9]. In fact, as a purely functional analysis argument, the [Ma2, Theorem 4.5, 4.6 and 4.7] can be extended formally to the equivariant case by introducing in the right place the operator g which stands for a fixed generator of $\mu_n(\mathbb{C})$ in Ma's article. The reason one can do this formal extension has been given in [Ma1, Section 5]. For the equivariant extensions of [Ma2, Theorem 4.8, 4.9, 4.10 and 4.11], one needs to show that their proofs are local on $f^{-1}(V_{\mu_n})$ and certain rescaling on Clifford variables which doesn't effect the action of g can be made (cf. [Ma2, Section 7 b)), this would guarantee that the main technic in [Ma2], the relative local index theorem can still be applied. While all necessary technics for doing the above equivariant extensions have been presented in [Ma1, Section 7, 8, and 9]. Hence we get the desired identity, details are left to interested readers. \square

Remark 3.2. Denote by $\Delta(f, h, \omega^W, \omega^V, \overline{E})$ the differential form which measures the difference

$$\begin{aligned} T_g(g, \omega^W, h^E) - T_g(h, \omega^V, h^{f_*E}) - \frac{1}{(2\pi i)^{r_h}} \int_{V_{\mu_n}/S} \text{Td}_g(\overline{Th}) T_g(f, \omega^W, h^E) \\ - \text{ch}_g(\overline{E}(f, h, h \circ f)) + \frac{1}{(2\pi i)^{r_g}} \int_{W_{\mu_n}/S} \text{Td}_g(\overline{T}(f, h, h \circ f)) \text{ch}_g(\overline{E}) \end{aligned}$$

in Theorem 3.1. Assume that we are in the same situation described before Lemma 2.8. Call $l : S \times Z_1 \rightarrow S \times Z$ the natural inclusion, then similar to Lemma 2.8, we have that

$$l^* \Delta(f_Z, h_Z, \omega^W, \omega^V, \overline{E}) = \Delta(f_{Z_1}, h_{Z_1}, \omega_1^W, \omega_1^V, j^* \overline{E}).$$

3.2 The transitivity property

In this subsection, we present certain transitivity property of direct image maps between equivariant higher arithmetic K-groups. To do this, we firstly write down the following diagram of homological complexes

$$\begin{array}{ccc}
 \widetilde{\mathbb{Z}}C_*(X, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n} \\
 \downarrow f_* & & \downarrow f_{\mu_n*} \circ \text{Td}_g(\overline{Tf}) \bullet (\cdot) \\
 \widetilde{\mathbb{Z}}C_*(Y, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(Y_{\mu_n}, p)_{R_n} \\
 \downarrow h_* & & \downarrow h_{\mu_n*} \circ \text{Td}_g(\overline{Th}) \bullet (\cdot) \\
 \widetilde{\mathbb{Z}}C_*(Z, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(Z_{\mu_n}, p)_{R_n}.
 \end{array} \tag{10}$$

As in last subsection, set $g = h \circ f$. Let \overline{E} be a hermitian k -cube in $\widehat{\mathcal{P}}(X, \mu_n)$, then the short exact sequence

$$0 \longrightarrow h_* f_* \overline{E} \xrightarrow{\text{Id}} g_* \overline{E} \longrightarrow 0 \longrightarrow 0$$

can be regarded as a hermitian $(k+1)$ -cube $H_{h \circ f}(\overline{E})$ on Z such that the transgression bundle $\text{tr}_{k+1}(\lambda(H_{h \circ f}(\overline{E})))$ satisfies the relation

$$\text{tr}_{k+1}(\lambda(H_{h \circ f}(\overline{E})))|_{Z \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_k(\lambda(g_* \overline{E})),$$

$$\text{tr}_{k+1}(\lambda(H_{h \circ f}(\overline{E})))|_{Z \times \{\infty\} \times (\mathbb{P}^1)^k} = \text{tr}_k(\lambda(h_* f_* \overline{E}))$$

and

$$\text{tr}_{k+1}(\lambda(H_{h \circ f}(\overline{E})))|_{Z \times (\mathbb{P}^1)^i \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_k(\lambda(H_{h \circ f}(\partial_i^0 \overline{E}))),$$

$$\text{tr}_{k+1}(\lambda(H_{h \circ f}(\overline{E})))|_{Z \times (\mathbb{P}^1)^i \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_k(\lambda(H_{h \circ f}(\partial_i^{-1} \overline{E}))) \oplus \text{tr}_k(\lambda(H_{h \circ f}(\partial_i^1 \overline{E})))$$

for $i = 1, \dots, k$.

Proposition 3.3. *The following map*

$$\Pi_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(H_{h \circ f}(\overline{E}))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2)$$

which vanishes on degenerate cubes provides a chain homotopy of homological complexes between the maps $\text{ch}_g \circ g_*$ and $\text{ch}_g \circ (h_* \circ f_*)$.

Proof. Using the above relations that the transgression bundle $\text{tr}_{k+1}(\lambda(H_{h \circ f}(\overline{E})))$ satisfies and the expression of dC_{k+1} , the proof is straightforward. This can be also seen from the fact that $H_{h \circ f}(\overline{E})$ provides a chain homotopy between g_* and $h_* \circ f_*$. \square

Proposition 3.4. *The composition $h_{\mu_n*} \circ \mathrm{Td}_g(\overline{Th}) \bullet (f_{\mu_n*} \circ \mathrm{Td}_g(\overline{Tf}) \bullet (\cdot))$ is equal to $g_{\mu_n*} \circ f_{\mu_n}^* \mathrm{Td}_g(\overline{Th}) \mathrm{Td}_g(\overline{Tf}) \bullet (\cdot)$. The following maps*

$$\Pi_k^{(3)}(\overline{E}) := \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left[\frac{1}{(2\pi i)^{r_g}} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Z_{\mu_n} \times (\mathbb{P}^1)^k} \mathrm{Td}_g(\overline{T}(f, h, h \circ f)) \right. \\ \left. \mathrm{ch}_g^0(\mathrm{tr}_k \circ \lambda(\overline{E})) \right] \bullet C_k(\log |z_1|^2, \dots, \log |z_k|^2)$$

and

$$\Pi_k^{(3')}(\overline{E}) := \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\frac{1}{(2\pi i)^{r_g}} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Z_{\mu_n} \times (\mathbb{P}^1)^k} \mathrm{Td}_g(\overline{T}(f, h, h \circ f)) \right. \\ \left. \mathrm{ch}_g^0(\mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2 \right)$$

give two chain homotopies of homological complexes between the maps $g_{\mu_n*} \circ \mathrm{Td}_g(\overline{Tg}) \bullet (\mathrm{ch}_g(\cdot))$ and $g_{\mu_n*} \circ f_{\mu_n}^* \mathrm{Td}_g(\overline{Th}) \mathrm{Td}_g(\overline{Tf}) \bullet (\mathrm{ch}_g(\cdot))$. Moreover, $\Pi_k^{(3)}(\overline{E})$ and $\Pi_k^{(3')}(\overline{E})$ are naturally homotopic to each other.

Proof. The first statement follows from the projection formula, the second statement follows from a straightforward computation and the third follows from [T3, Remark 2.4, Lemma. 2.5]. \square

Now we write $\Pi_k^f = \Pi_k'^f + \Pi_k''^f$ for the chain homotopy of the upper square in (10) and $\Pi_k^h = \Pi_k'^h + \Pi_k''^h$ for the chain homotopy of the lower square in (10). Then $\Pi_k^{(1)} + h_{\mu_n*} \circ (\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k^f) + \Pi_k^h \circ f_* - \Pi_k^{(3)}$ decides a chain homotopy of homological complex between maps $\mathrm{ch}_g \circ g_*$ and $g_{\mu_n*} \circ \mathrm{Td}_g(\overline{Tg}) \bullet (\mathrm{ch}_g(\cdot))$. Suppose that the μ_n -action on Z is trivial, it's the main result of this subsection that the chain homotopy $\Pi_k^{(1)} + h_{\mu_n*} \circ (\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k^f) + \Pi_k^h \circ f_* - \Pi_k^{(3)}$ is homotopic to the chain homotopy $\Pi_k^g = \Pi_k'^g + \Pi_k''^g$ for the whole square in (10). According to Proposition 3.4, it is equivalent to show that $\Pi_k^{(1)} + h_{\mu_n*} \circ (\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k^f) + \Pi_k^h \circ f_* - \Pi_k^{(3)}$ is homotopic to Π_k^g .

To see this, we firstly denote by $H_{hof}^g(\overline{E})$ the following emi-2-cube of hermitian bundles on $Z \times (\mathbb{P}^1)^k$

$$\begin{array}{ccccc} h_* f_*(\mathrm{tr}_k \circ \lambda(\overline{E})) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda(h_* f_*(\overline{E})) & \longrightarrow & 0 \\ \downarrow \mathrm{Id} & & \downarrow \mathrm{Id} & & \downarrow \\ g_*(\mathrm{tr}_k \circ \lambda(\overline{E})) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda(g_*(\overline{E})) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Then, like before, we construct a hermitian bundle $\mathrm{tr}_2(H_{hof}^g(\overline{E}))$ on $(\mathbb{P}^1)^{k+2}$ as a modified second transgression bundle of $H_{hof}^g(\overline{E})$ such that it satisfies the following relations:

$$\mathrm{tr}_2(H_{hof}^g(\overline{E}))|_{Z \times \{0\} \times (\mathbb{P}^1)^{k+1}} = \mathrm{tr}_{k+1}(\lambda(H_{hof}(\overline{E}))),$$

$$\begin{aligned}\mathrm{tr}_2(H_{hof}^g(\overline{E}))|_{Z \times \{\infty\} \times (\mathbb{P}^1)^{k+1}} &= \mathrm{tr}_1(H_{hof}(\mathrm{tr}_k \circ \lambda(\overline{E}))), \\ \mathrm{tr}_2(H_{hof}^g(\overline{E}))|_{Z \times \mathbb{P}^1 \times \{0\} \times (\mathbb{P}^1)^k} &= \mathrm{tr}_1(H(\overline{E}, g_*)), \\ \mathrm{tr}_2(H_{hof}^g(\overline{E}))|_{Z \times \mathbb{P}^1 \times \{\infty\} \times (\mathbb{P}^1)^k} &= \mathrm{tr}_1(H(\overline{E}, h_* f_*))\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}_2(H_{hof}^g(\overline{E}))|_{Z \times (\mathbb{P}^1)^{i+1} \times \{0\} \times (\mathbb{P}^1)^{k-i}} &= \mathrm{tr}_2(H_{hof}^g(\partial_i^0 \overline{E})), \\ \mathrm{tr}_2(H_{hof}^g(\overline{E}))|_{Z \times (\mathbb{P}^1)^{i+1} \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} &= \mathrm{tr}_2(H_{hof}^g(\partial_i^{-1} \overline{E})) \oplus \mathrm{tr}_2(H_{hof}^g(\partial_i^1 \overline{E}))\end{aligned}$$

for $i = 1, \dots, k$. We set

$$\mathbf{H}_{1,k}(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0(\mathrm{tr}_2(H_{hof}^g(\overline{E}))) \wedge C_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2).$$

Then $\mathbf{H}_{1,k}$ vanishes on degenerate k -cubes, and we obtain a map

$$\mathbf{H}_{1,k} : \widetilde{\mathbb{Z}}C_k(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} D^{2p-k-2}(Z, p)_{R_n}$$

by linear extension. This map satisfies the following differential equation

$$\begin{aligned}d \circ \mathbf{H}_{1,k}(\overline{E}) &= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0(\mathrm{tr}_2(H_{hof}^g(\overline{E}))) \wedge dC_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2) \\ &= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0(\mathrm{tr}_2(H_{hof}^g(\overline{E}))) \wedge [(-\frac{1}{2})(k+2) \sum_{j=1}^{k+2} (-1)^{j-1} (-4\pi i) \\ &\quad (\delta_{z_j=\infty} - \delta_{z_j=0}) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_j|^2, \dots, \log |z_{k+2}|^2)] \\ &= \mathbf{H}_{1,k} \circ d(\overline{E}) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} [\mathrm{ch}_g^0(\mathrm{tr}_1(H(\overline{E}, g_*))) - \\ &\quad \mathrm{ch}_g^0(\mathrm{tr}_1(H(\overline{E}, h_* f_*)))] \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &\quad + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} [\mathrm{ch}_g^0(\mathrm{tr}_{k+1}(\lambda(H_{hof}(\overline{E})))) - \\ &\quad \mathrm{ch}_g^0(\mathrm{tr}_1(H_{hof}(\mathrm{tr}_k \circ \lambda(\overline{E}))))] \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &= \mathbf{H}_{1,k} \circ d(\overline{E}) - \Pi_k'^g(\overline{E}) + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0(\mathrm{tr}_1(H(\overline{E}, h_* f_*))) \wedge \\ &\quad C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &\quad + \Pi_k^{(1)}(\overline{E}) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0(\mathrm{tr}_1(H_{hof}(\mathrm{tr}_k \circ \lambda(\overline{E})))) \wedge \\ &\quad C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2).\end{aligned}$$

Secondly, we denote by $H'_{hof}{}^g(\overline{E})$ the following emi-2-cube of hermitian bundles on $Z \times (\mathbb{P}^1)^k$

$$\begin{array}{ccccc}
 h_* f_*(\mathrm{tr}_k \circ \lambda(\overline{E})) & \xrightarrow{\mathrm{Id}} & h_* \mathrm{tr}_k \circ \lambda(f_*(\overline{E})) & \longrightarrow & 0 \\
 \downarrow \mathrm{Id} & & \downarrow \mathrm{Id} & & \downarrow \\
 h_* f_*(\mathrm{tr}_k \circ \lambda(\overline{E})) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda(h_* f_*(\overline{E})) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & 0.
 \end{array}$$

Again, we construct a hermitian bundle $\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))$ on $Z \times (\mathbb{P}^1)^{k+2}$ as a modified second transgression bundle of $H'_{hof}{}^g(\overline{E})$ such that it satisfies the following relations:

$$\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))|_{Z \times \{0\} \times (\mathbb{P}^1)^{k+1}} = \mathrm{tr}_1(H(f_* \overline{E}, h_*)),$$

$$\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))|_{Z \times \{\infty\} \times (\mathbb{P}^1)^{k+1}} = \mathrm{tr}_1(h_* f_*(\mathrm{tr}_k \circ \lambda(\overline{E})) \rightarrow h_* f_*(\mathrm{tr}_k \circ \lambda(\overline{E}))),$$

$$\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))|_{Z \times \mathbb{P}^1 \times \{0\} \times (\mathbb{P}^1)^k} = \mathrm{tr}_1(H(\overline{E}, h_* f_*)),$$

$$\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))|_{Z \times \mathbb{P}^1 \times \{\infty\} \times (\mathbb{P}^1)^k} = \mathrm{tr}_1(h_* H(\overline{E}, f_*))$$

and

$$\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))|_{Z \times (\mathbb{P}^1)^{i+1} \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \mathrm{tr}_2(H'_{hof}{}^g(\partial_i^0 \overline{E})),$$

$$\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))|_{Z \times (\mathbb{P}^1)^{i+1} \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \mathrm{tr}_2(H'_{hof}{}^g(\partial_i^{-1} \overline{E})) \oplus \mathrm{tr}_2(H'_{hof}{}^g(\partial_i^1 \overline{E}))$$

for $i = 1, \dots, k$. We set

$$\mathbf{H}_{2,k}(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0(\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))) \wedge C_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2).$$

Then $\mathbf{H}_{2,k}$ decides a map

$$\mathbf{H}_{2,k} : \widetilde{\mathbb{Z}}C_k(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} D^{2p-k-2}(Z, p)_{R_n}$$

which satisfies the following differential equation

$$\begin{aligned}
 d \circ \mathbf{H}_{2,k}(\overline{E}) &= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0(\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))) \wedge dC_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2) \\
 &= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0(\mathrm{tr}_2(H'_{hof}{}^g(\overline{E}))) \wedge [(-\frac{1}{2})(k+2) \sum_{j=1}^{k+2} (-1)^{j-1} (-4\pi i) \\
 &\quad (\delta_{z_j=\infty} - \delta_{z_j=0}) \wedge C_{k+1}(\log |z_1|^2, \dots, \widehat{\log |z_j|^2}, \dots, \log |z_{k+2}|^2)] \\
 &= \mathbf{H}_{2,k} \circ d(\overline{E}) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} [\mathrm{ch}_g^0(\mathrm{tr}_1(H(\overline{E}, h_* f_*))) -
 \end{aligned}$$

$$\begin{aligned}
& \text{ch}_g^0(\text{tr}_1(h_*H(\overline{E}, f_*))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
& + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H(f_*\overline{E}, h_*))) \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
& = \mathbf{H}_{2,k} \circ d(\overline{E}) + \Pi_k^h(f_*\overline{E}) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H(\overline{E}, h_*f_*))) \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
& + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(h_*H(\overline{E}, f_*))) \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2).
\end{aligned}$$

Thirdly, notice that the short exact sequence

$$h_* \text{tr}_1(H(\overline{E}, f_*)) \xrightarrow{\text{Id}} \text{tr}_1(h_*H(\overline{E}, f_*)) \longrightarrow 0$$

forms an emi-1-cube of hermitian bundles on $Z \times \mathbb{P}^1 \times (\mathbb{P}^1)^k$, we denote it by $\tilde{H}_{hof}(\overline{E})$. Using the same construction as before, we construct a transgression bundle $\text{tr}_1(\tilde{H}_{hof}(\overline{E}))$ on $Z \times \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1)^k$ satisfying

$$\text{tr}_1(\tilde{H}_{hof}(\overline{E}))|_{Z \times \{0\} \times (\mathbb{P}^1)^{k+1}} = \text{tr}_1(h_*H(\overline{E}, f_*)),$$

$$\text{tr}_1(\tilde{H}_{hof}(\overline{E}))|_{Z \times \{\infty\} \times (\mathbb{P}^1)^{k+1}} = h_* \text{tr}_1(H(\overline{E}, f_*)),$$

$$\text{tr}_1(\tilde{H}_{hof}(\overline{E}))|_{Z \times \mathbb{P}^1 \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_1(h_* \text{tr}_k \circ \lambda(f_*\overline{E}) \rightarrow h_* \text{tr}_k \circ \lambda(f_*\overline{E}) \rightarrow 0),$$

$$\text{tr}_1(\tilde{H}_{hof}(\overline{E}))|_{Z \times \mathbb{P}^1 \times \{\infty\} \times (\mathbb{P}^1)^k} = \text{tr}_1(h_* f_* \text{tr}_k \circ \lambda(\overline{E}) \rightarrow h_* f_* \text{tr}_k \circ \lambda(\overline{E}) \rightarrow 0)$$

and

$$\text{tr}_1(\tilde{H}_{hof}(\overline{E}))|_{Z \times (\mathbb{P}^1)^{i+1} \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_1(\tilde{H}_{hof}(\partial_i^0 \overline{E})),$$

$$\text{tr}_1(\tilde{H}_{hof}(\overline{E}))|_{Z \times (\mathbb{P}^1)^{i+1} \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_1(\tilde{H}_{hof}(\partial_i^{-1} \overline{E})) \oplus \text{tr}_1(\tilde{H}_{hof}(\partial_i^1 \overline{E}))$$

for $i = 1, \dots, k$. So if we set

$$\mathbf{H}_{3,k}(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}_g^0(\text{tr}_1(\tilde{H}_{hof}(\overline{E}))) \wedge C_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2),$$

it satisfies the differential equation

$$\begin{aligned}
d \circ \mathbf{H}_{3,k}(\overline{E}) &= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}_g^0(\text{tr}_1(\tilde{H}_{hof})) \wedge dC_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2) \\
&= \mathbf{H}_{3,k} \circ d(\overline{E}) + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(h_*H(\overline{E}, f_*))) \wedge
\end{aligned}$$

$$\begin{aligned}
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
& - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(h_* \text{tr}_1(H(\overline{E}, f_*))) \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2)
\end{aligned}$$

Finally, we set

$$\mathbf{H}_{4,k}(\overline{E}) := \frac{(-1)^{k+2}}{(k+2)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} C_{k+2}(T_g(h, h^{\text{tr}_1(H(\overline{E}, f_*)))), \log |z_1|^2, \dots, \log |z_{k+1}|^2),$$

then it satisfies

$$\begin{aligned}
d \circ \mathbf{H}_{4,k}(\overline{E}) &= \frac{(-1)^{k+2}}{(k+2)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} C_{k+2}(T_g(h, h^{\text{tr}_1(H(\overline{E}, f_*)))), \log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
&= \mathbf{H}_{4,k} \circ d(\overline{E}) + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(h_* \text{tr}_1(H(\overline{E}, f_*))) \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
& - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \left[\frac{1}{(2\pi i)^{r_h}} \int_{Y_{\mu_n}} \text{Td}_g(\overline{Th}) \text{ch}_g^0(\text{tr}_1(H(\overline{E}, f_*))) \right] \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
& - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(h, h^{\text{tr}_k \circ \lambda(f_* \overline{E})}), \log |z_1|^2, \dots, \log |z_k|^2) \\
& + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(h, h^{f_* \text{tr}_k \circ \lambda(\overline{E})}), \log |z_1|^2, \dots, \log |z_k|^2) \\
&= \mathbf{H}_{4,k} \circ d(\overline{E}) - h_{\mu_n *} \circ (\text{Td}_g(\overline{Th}) \bullet \Pi_k^f(\overline{E})) \\
& + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(h_* \text{tr}_1(H(\overline{E}, f_*))) \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) - \Pi_k^{fh}(f_* \overline{E}) \\
& + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(h, h^{f_* \text{tr}_k \circ \lambda(\overline{E})}), \log |z_1|^2, \dots, \log |z_k|^2)
\end{aligned}$$

Proposition 3.5. *Let notations and assumptions be as above, then the chain homotopy $\Pi_k^g = \Pi_k'^g + \Pi_k''^g$ is homotopic to $\Pi_k^{(1)} + h_{\mu_n *} \circ (\text{Td}_g(\overline{Th}) \bullet \Pi_k^f) + \Pi_k^h \circ f_* - \Pi_k^{(3')}$.*

Proof. Let \overline{E} be a hermitian k -cube in $\widehat{\mathcal{P}}(X, \mu_n)$ which is f -acyclic and g -acyclic. Using the above differential equations concerning $\mathbf{H}_{i,k}$, we obtain that

$$\begin{aligned}
& (\mathbf{H}_{1,k} + \mathbf{H}_{2,k} - \mathbf{H}_{3,k} - \mathbf{H}_{4,k}) \circ d(\overline{E}) - d \circ (\mathbf{H}_{1,k} + \mathbf{H}_{2,k} - \mathbf{H}_{3,k} - \mathbf{H}_{4,k})(\overline{E}) \\
&= \Pi_k'^g(\overline{E}) - \Pi_k^{(1)}(\overline{E}) - \Pi_k^{fh}(f_* \overline{E}) - \Pi_k^{fh}(f_* \overline{E}) - h_{\mu_n *} \circ (\text{Td}_g(\overline{Th}) \bullet \Pi_k^f(\overline{E}))
\end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H_{h \circ f}(\text{tr}_k \circ \lambda(\overline{E})))) \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
& + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(h, h^{f_* \text{tr}_k \circ \lambda(\overline{E})}), \log |z_1|^2, \dots, \log |z_k|^2)
\end{aligned}$$

On the other hand, according to Theorem 3.1, we have

$$\begin{aligned}
& \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(g, h^{\text{tr}_k \circ \lambda(\overline{E})}), \log |z_1|^2, \dots, \log |z_k|^2) \\
& - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(h, h^{f_* \text{tr}_k \circ \lambda(\overline{E})}), \log |z_1|^2, \dots, \log |z_k|^2) \\
& - h_{\mu_n *} \circ (\text{Td}_g(\overline{T}h) \bullet \Pi_k''^f(\overline{E})) \\
& = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H_{h \circ f}(\text{tr}_k \circ \lambda(\overline{E})))) \wedge \\
& C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) - \Pi_k^{(3')}(\overline{E}) \\
& + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d\Delta(f, h, \omega^X, \omega^Y, \text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2).
\end{aligned}$$

We then formally define a product $C_{k+1}(\Delta(f, h, \omega^X, \omega^Y, \text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2)$ in the same way as (9), and we set

$$\Delta_k(\overline{E}) = \frac{(-1)^k}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(\Delta(f, h, \omega^X, \omega^Y, \text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2).$$

It is readily checked by Lemma 3.2 that

$$\begin{aligned}
& \Delta_{k-1}(d\overline{E}) - d\Delta_k(\overline{E}) \\
& = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d\Delta(f, h, \omega^X, \omega^Y, \text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2).
\end{aligned}$$

Combing all the above computations, we finally get

$$\begin{aligned}
& (\mathbf{H}_{1,k-1} + \mathbf{H}_{2,k-1} - \mathbf{H}_{3,k-1} - \mathbf{H}_{4,k-1} + \Delta_{k-1})(d\overline{E}) - d(\mathbf{H}_{1,k} + \mathbf{H}_{2,k} - \mathbf{H}_{3,k} - \mathbf{H}_{4,k} + \Delta_k)(\overline{E}) \\
& = \Pi_k'^g(\overline{E}) - \Pi_k^{(1)}(\overline{E}) - \Pi_k'^h(f_*\overline{E}) - \Pi_k''^h(f_*\overline{E}) - h_{\mu_n *} \circ (\text{Td}_g(\overline{T}h) \bullet \Pi_k'^f(\overline{E})) \\
& \quad + \Pi_k''^g(\overline{E}) - h_{\mu_n *} \circ (\text{Td}_g(\overline{T}h) \bullet \Pi_k''^f(\overline{E})) + \Pi_k^{(3')}(\overline{E}) \\
& = \Pi_k^g(\overline{E}) - (\Pi_k^{(1)}(\overline{E}) + h_{\mu_n *} \circ (\text{Td}_g(\overline{T}h) \bullet \Pi_k^f(\overline{E}))) + \Pi_k^h(f_*\overline{E}) - \Pi_k^{(3')}(\overline{E})
\end{aligned}$$

So we are done. \square

Corollary 3.6. *Let $f : X \rightarrow Y$, $h : Y \rightarrow Z$ and $g : X \rightarrow Z$ be three equivariant and proper morphisms between μ_n -equivariant schemes, which are all smooth over the generic fibres. Assume that $g = h \circ f$ and that the μ_n -action on Z is trivial. Then the direct image map g_* is equal to the composition $h_* \circ f_*$ from $\widehat{K}_m(X, \mu_n)_{\mathbb{Q}}$ to $\widehat{K}_m(Z, \mu_n)_{\mathbb{Q}}$ for any $m \geq 1$.*

4 The Lefschetz-Riemann-Roch theorem

4.1 The statement

In order to formulate the Lefschetz-Riemann-Roch theorem for higher equivariant arithmetic K-groups, we need to introduce the equivariant R -genus due to Bismut. Let X be a μ_n -equivariant smooth algebraic variety over \mathbb{C} , and let \overline{E} be a μ_n -equivariant hermitian vector bundle on X . For $\zeta \in \mu_n(\mathbb{C})$ and $s > 1$, we consider the following Lerch zeta function

$$L(\zeta, s) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^s}$$

and its meromorphic continuation to the whole complex plane. Define a formal power series in the variable x as

$$\tilde{R}(\zeta, x) := \sum_{n=0}^{\infty} \left(\frac{\partial L}{\partial s}(\zeta, -n) + L(\zeta, -n) \sum_{j=1}^n \frac{1}{2j} \right) \frac{x^n}{n!}.$$

Definition 4.1. The Bismut's equivariant R -genus of an equivariant hermitian vector bundle \overline{E} with $\overline{E}|_{X_{\mu_n}} = \sum_{\zeta \in \mu_n(\mathbb{C})} \overline{E}_{\zeta}$ is defined as

$$R_g(\overline{E}) := \sum_{\zeta \in \mu_n(\mathbb{C})} (\mathrm{Tr} \tilde{R}(\zeta, -\Omega^{\overline{E}_{\zeta}}) - \mathrm{Tr} \tilde{R}(1/\zeta, \Omega^{\overline{E}_{\zeta}})),$$

where $\Omega^{\overline{E}_{\zeta}}$ is the curvature form associated to \overline{E}_{ζ} .

Now, let X be a μ_n -equivariant arithmetic scheme over an arithmetic ring (D, Σ, F_{∞}) and we construct a naive commutative diagram of homological complexes

$$\begin{array}{ccc} \tilde{\mathbb{Z}}C_*(X, \mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n} \\ \downarrow 0 & & \downarrow 0 \\ \tilde{\mathbb{Z}}C_*(X, \mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n} \end{array} \quad (11)$$

where 0 stands for the zero map. Let \overline{N} be a μ_n -equivariant hermitian vector bundle on X , we shall regard the R -genus $R_g(\overline{N})$ as an element in $\bigoplus_{p \geq 0} D^{2p-1}(X, p)$. It is a d -closed form. Denote by p_0 the natural projection from $X \times (\mathbb{P}^1)^{\cdot}$ to X . For any hermitian k -cube \overline{E} in $\widehat{\mathcal{P}}(X, \mu_n)$, we set

$$\Pi_R(\overline{E}) = \frac{(-1)^k}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(R_g(p_0^* \overline{N}) \mathrm{ch}_g^0(\mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2).$$

It is clear that $\Pi_R(\overline{E})$ extends to be a map $\Pi_R : \tilde{\mathbb{Z}}C_k(X, \mu_n) \rightarrow \bigoplus_{p \geq 0} D^{2p-k-1}(X_{\mu_n}, p)_{R_n}$ which provides a chain homotopy for the square (11). Therefore, we get an endomorphism of $\hat{K}_m(X, \mu_n)$ for any $m \geq 1$. This endomorphism will be denoted by $\otimes R_g(\overline{N})$.

Again, by [T3, Remark 2.4, Lemma 2.5], the chain homotopy Π_R is homotopic to the chain homotopy Π'_R defined by

$$\Pi'_R(\overline{E}) = \frac{(-1)^{k+1}}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} R_g(p_0^* \overline{N}) \bullet \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E})) \bullet C_k(\log |z_1|^2, \dots, \log |z_k|^2),$$

and hence is homotopic to $-R_g(\overline{N}) \bullet \text{ch}_g(\overline{E})$ by the projection formula. Let (x, α) be an element in $\widehat{K}_m(X, \mu_n)_{\mathbb{Q}}$, then $dx = 0$ and $\text{ch}_g(x)$ is a d -closed form. Let $(0, \alpha)$ and $(0, \alpha')$ be two elements in $\widehat{K}_m(X, \mu_n)_{\mathbb{Q}}$, then $(0, \alpha) = (0, \alpha')$ if α and α' have the same cohomology class in $\bigoplus_{p \geq 0} H_{\mathcal{D}}^*(X_{\mu_n}, \mathbb{R}(p))_{R_n}$. Notice that the product \bullet on the Deligne-Beilinson complex induces the product on the real Deligne-Beilinson cohomology. Then, modulo torsion, the endomorphism $\otimes R_g(\overline{N})$ is independent of the choice of the metric on N and it can be written as $\otimes R_g(N)$.

Let X_{μ_n} be the fixed point subscheme of X , and let $\overline{N}_{X/X_{\mu_n}}$ be the normal bundle of X_{μ_n} in X with some μ_n -invariant hermitian metric. We set

$$\Lambda_R := (\text{Id} - \otimes R_g(N_{X/X_{\mu_n}})) \circ \otimes \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^{\vee}),$$

it is a well-defined endomorphism of $\widehat{K}_m(X_{\mu_n}, \mu_n)_{\rho} \otimes \mathbb{Q}$. Then the arithmetic Lefschetz-Riemann-Roch theorem for higher equivariant arithmetic K-groups can be formulated as follows.

Theorem 4.2. (*arithmetic Lefschetz-Riemann-Roch*) *Let $f : X \rightarrow Y$ be an equivariant and proper morphism between two μ_n -equivariant arithmetic schemes, which is smooth over generic fibre. Suppose that the μ_n -action on the base Y is trivial. Then, for any $m \geq 1$, the following diagram*

$$\begin{array}{ccc} \widehat{K}_m(X, \mu_n) & \xrightarrow{\Lambda_R \circ \tau} & \widehat{K}_m(X_{\mu_n}, \mu_n)_{\rho} \otimes \mathbb{Q} \\ f_* \downarrow & & \downarrow f_{\mu_n *} \\ \widehat{K}_m(Y, \mu_n) & \longrightarrow & \widehat{K}_m(Y, \mu_n)_{\rho} \otimes \mathbb{Q} \end{array}$$

where τ is the restriction map, is commutative.

The proof of Theorem 4.2 will be given in next two subsections.

4.2 Arithmetic K-theoretic form of Bismut-Ma's immersion formula

Let $Y \hookrightarrow X$ be a μ_n -equivariant closed immersion of regular μ_n -projective arithmetic schemes over (D, Σ, F_{∞}) . In [T3, Section 4], we have proved an arithmetic purity theorem

$$\widehat{K}_m(Y, \mu_n) \cong \widehat{K}_{Y,m}(X, \mu_n)$$

for any integer $m \geq 1$. As a byproduct, we get an embedding morphism $\widehat{K}_m(Y, \mu_n) \rightarrow \widehat{K}_m(X, \mu_n)$. This embedding morphism is realized by constructing an explicit chain homotopy

for the square

$$\begin{array}{ccc} \tilde{\mathbb{Z}}C_*(Y, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} {}'D^{2p-*}(Y_{\mu_n}, p)_{R_n} \\ \downarrow i_* & & \downarrow i_{\mu_n!} \circ \text{Td}_g^{-1}(\overline{N}_{X/Y}) \bullet (\cdot) \\ \tilde{\mathbb{Z}}C_*(P, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} {}'D^{2p-*}(P_{\mu_n}, p)_{R_n}, \end{array} \quad (12)$$

where $'D^{2p-*}(\cdot, p)$ stands for the Dolbeaut complex computing the Deligne homology groups, $i : Y \hookrightarrow P := \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ is the associated zero section embedding with projection $\pi : P \rightarrow Y$ and

$$i_* : \tilde{\mathbb{Z}}C_*(Y, \mu_n) \rightarrow \tilde{\mathbb{Z}}C_*(P, \mu_n)$$

is the morphism of homological complexes by sending a hermitian cube \overline{E} to $\sum_{j=0}^n (-1)^j \overline{Q}^\vee \otimes \pi^* \overline{E}$ provided the Koszul resolution

$$K(\overline{E}, \overline{N}_{X/Y}) : 0 \rightarrow \wedge^n \overline{Q}^\vee \otimes \pi^* \overline{E} \rightarrow \cdots \rightarrow \wedge \overline{Q}^\vee \otimes \pi^* \overline{E} \rightarrow \pi^* \overline{E} \rightarrow i_* \overline{E} \rightarrow 0.$$

For any hermitian k -cube \overline{E} , the chain homotopy $\mathbf{H}_k(\overline{E})$ for (12) is given by the formula

$$\mathbf{H}_k(\overline{E}) = T_g(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y}) \bullet \text{ch}_g(\overline{E}))$$

where $T_g(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y}))$ is the equivariant Bott-Chern singular current associated to the canonical Koszul resolution. Notice that the product $P \times (\mathbb{P}^1)^\cdot$ can be identified with the projective space bundle over $Y \times (\mathbb{P}^1)^\cdot$ with respect to the vector bundle $p_0^* N_{X/Y}$, and

$$0 \rightarrow p_0^* \wedge^n \overline{Q}^\vee \rightarrow \cdots \rightarrow p_0^* \wedge \overline{Q}^\vee \rightarrow \overline{\mathcal{O}}_{P \times (\mathbb{P}^1)^\cdot} \rightarrow i_* \overline{\mathcal{O}}_{Y \times (\mathbb{P}^1)^\cdot} \rightarrow 0$$

represents the Koszul resolution so that the corresponding Bott-Chern singular current is the pullback $p_0^* T_g(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y}))$. We shall still write it as $T_g(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y}))$ for the sake of simplicity. Then, like before, by the projection formula and [T3, Remark 2.4, Lemma 2.5], $\mathbf{H}_k(\overline{E})$ is naturally homotopic to the following chain homotopy

$$\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y}) \bullet \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2),$$

which will be still denoted by $\mathbf{H}_k(\overline{E})$.

It is clear that if we choose another resolution

$$0 \rightarrow \overline{F}_n \rightarrow \cdots \rightarrow \overline{F}_1 \rightarrow \overline{F}_0 \rightarrow i_* \overline{\mathcal{O}}_Y \rightarrow 0$$

with respect to the zero section embedding $i : Y \hookrightarrow \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ such that the metrics on F satisfy the Bismut's assumption (A), we may construct a different homotopy for (12) and we shall get another embedding morphism $i_* : \hat{K}_m(Y, \mu_n) \rightarrow \hat{K}_m(P, \mu_n)$. Our first result in this subsection is the following.

Proposition 4.3. *The embedding morphism over rational arithmetic K-groups*

$$i_* : \widehat{K}_m(Y, \mu_n)_{\mathbb{Q}} \rightarrow \widehat{K}_m(P, \mu_n)_{\mathbb{Q}}$$

is independent of the choice of the resolution of $i_*\overline{\mathcal{O}}_Y$ on $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$.

Proof. Since any two resolutions of $i_*\overline{\mathcal{O}}_Y$ on $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ are dominated by a third one, we may assume that \overline{F}_\bullet and $\wedge^\bullet \overline{Q}^\vee$ fit into the following diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \overline{A}_n & \longrightarrow & \overline{F}_n & \longrightarrow & \wedge^n \overline{Q}^\vee \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots \\
 0 & \longrightarrow & \overline{A}_1 & \longrightarrow & \overline{F}_1 & \longrightarrow & \wedge^1 \overline{Q}^\vee \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \overline{A}_0 & \longrightarrow & \overline{F}_0 & \longrightarrow & \overline{\mathcal{O}}_P \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & i_*\overline{\mathcal{O}}_Y & \longrightarrow & i_*\overline{\mathcal{O}}_Y
 \end{array}$$

where \overline{A}_\bullet is an exact sequence of hermitian vector bundles on P . We may require that \overline{A}_\bullet is orthogonally split, then we split \overline{A}_\bullet into a family of short exact sequence of hermitian bundles from $j = 1$ to $n - 1$

$$\chi_j : \quad 0 \longrightarrow \text{Ker} d_j \longrightarrow \overline{A}_j \xrightarrow{d_j} \text{Ker} d_{j-1} \longrightarrow 0$$

such that every χ_j is orthogonally split. Moreover, we denote by ε_j the short exact sequence

$$0 \longrightarrow \overline{A}_j \longrightarrow \overline{F}_j \longrightarrow \wedge^j \overline{Q}^\vee \longrightarrow 0$$

from $j = 0$ to n . Write i_* (resp. i'_*) for the morphism $\widetilde{\mathbb{Z}}C_*(Y, \mu_n) \rightarrow \widetilde{\mathbb{Z}}C_*(P, \mu_n)$ with respect to the Koszul resolution $K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y})$ (resp. the resolution \overline{F}_\bullet). Then, for any hermitian k -cube \overline{E} on Y , the assignment

$$H_i(\overline{E}) := \sum_{j=0}^n (-1)^j \varepsilon_j \otimes \pi^* \overline{E} + \sum_{j=1}^{n-1} (-1)^j \chi_j \otimes \pi^* \overline{E} \in \widetilde{\mathbb{Z}}C_{k+1}(P, \mu_n)$$

provides a chain homotopy between i'_* and i_* . Consequently, the formula

$$\mathbf{H}_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(H_i(\overline{E}))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2)$$

decides a chain homotopy between $\text{ch}_g \circ i'_*$ and $\text{ch}_g \circ i_*$. We claim that there exists a homotopy of chain homotopies between $\mathbf{H}'_k(\overline{E})$ and $\mathbf{H}_k^{(1)}(\overline{E}) + \mathbf{H}_k(\overline{E})$. In fact, according to the construction of transgression bundles, we have that

$$\text{tr}_{k+1} \circ \lambda(\varepsilon_j \otimes \pi^* \overline{E}) = \text{tr}_1 \circ \lambda(\varepsilon_j) \boxtimes \text{tr}_k \circ \lambda(\pi^* \overline{E})$$

and

$$\text{tr}_{k+1} \circ \lambda(\chi_j \otimes \pi^* \overline{E}) = \text{tr}_1 \circ \lambda(\chi_j) \boxtimes \text{tr}_k \circ \lambda(\pi^* \overline{E}).$$

Moreover, the complex

$$0 \rightarrow \text{tr}_1 \circ \lambda(\varepsilon_n) \boxtimes \text{tr}_k \circ \lambda(\pi^* \overline{E}) \rightarrow \dots \rightarrow \text{tr}_1 \circ \lambda(\varepsilon_1) \boxtimes \text{tr}_k \circ \lambda(\pi^* \overline{E}) \rightarrow \text{tr}_1 \circ \lambda(\varepsilon_0) \boxtimes \text{tr}_k \circ \lambda(\pi^* \overline{E})$$

provides a resolution of $i_* \text{tr}_1(0 \rightarrow \text{tr}_k \circ \lambda(\pi^* \overline{E}) \rightarrow \text{tr}_k \circ \lambda(\pi^* \overline{E}))$ on $P \times (\mathbb{P}^1)^{k+1}$ satisfying Bismut's assumption (A) since the transgression functor is an exact functor. We denote this resolution by $\overline{\Xi}$. and we set

$$\tilde{\mathbf{H}}_k(\overline{E}) := \frac{(-1)^{k+2}}{(k+2)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} C_{k+2}(T_g(\overline{\Xi}), \log |z_1|^2, \dots, \log |z_{k+1}|^2),$$

then it satisfies

$$\begin{aligned} d \circ \tilde{\mathbf{H}}_k(\overline{E}) &= \frac{(-1)^{k+2}}{(k+2)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} C_{k+2}(T_g(\overline{\Xi}), \log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &= \tilde{\mathbf{H}}_k \circ d(\overline{E}) + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0 \left(\sum_{j=0}^n (-1)^j \text{tr}_1 \circ \lambda(\varepsilon_j) \boxtimes \text{tr}_k \circ \lambda(\pi^* \overline{E}) \right) \wedge \\ &\quad C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &\quad - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} i_{\mu_n}! [\text{ch}_g^0(\text{tr}_1(0 \rightarrow \text{tr}_k \circ \lambda(\pi^* \overline{E}) \rightarrow \text{tr}_k \circ \lambda(\pi^* \overline{E}))) \bullet \\ &\quad \text{Td}_g^{-1}(\overline{N}_{X/Y})] \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ &\quad - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(\overline{F}), \bullet \text{ch}_g^0(\text{tr}_k \circ \lambda(\pi^* \overline{E})), \log |z_1|^2, \dots, \log |z_k|^2) \\ &\quad + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y})), \bullet \text{ch}_g^0(\text{tr}_k \circ \lambda(\pi^* \overline{E})), \\ &\quad \log |z_1|^2, \dots, \log |z_k|^2) \\ &\quad + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\left[\frac{1}{4\pi i} \int_{(\mathbb{P}^1)^{k+1}/(\mathbb{P}^1)^k} \text{ch}_g^0 \left(\sum_{j=1}^{n-1} (-1)^j \text{tr}_1 \circ \lambda(\varepsilon_j) \right) \log |z_1|^2 \right] \bullet \right. \end{aligned}$$

$$\begin{aligned} & \text{ch}_g^0(\text{tr}_k \circ \lambda(\pi^* \bar{E})), \log |z_2|^2, \dots, \log |z_{k+1}|^2) \\ &= \tilde{\mathbf{H}}_k \circ d(\bar{E}) + \mathbf{H}_k^{(1)}(\bar{E}) - \mathbf{H}'_k(\bar{E}) + \mathbf{H}_k(\bar{E}) \end{aligned}$$

that is

$$\tilde{\mathbf{H}}_k \circ d(\bar{E}) - d \circ \tilde{\mathbf{H}}_k(\bar{E}) = \mathbf{H}'_k(\bar{E}) - (\mathbf{H}_k^{(1)}(\bar{E}) + \mathbf{H}_k(\bar{E})).$$

So we are done. \square

Now, let us recall the Bismut-Ma's immersion formula which relates analytic torsion forms and the Bott-Chern singular current. Let X be a μ_n -equivariant smooth algebraic variety over \mathbb{C} and let $i : Y \hookrightarrow X$ be an equivariant closed smooth subvariety. Let S be a smooth algebraic variety with trivial μ_n -action, and let $f : Y \rightarrow S$, $l : X \rightarrow S$ be two equivariant proper smooth morphisms such that $f = l \circ i$. Assume that $\bar{\eta}$ is an equivariant hermitian bundle on Y and $\bar{\xi}$ is a complex of equivariant hermitian bundles on X which provides a resolution of $i_* \bar{\eta}$ such that the metrics on ξ satisfy the Bismut's assumption (A). Let ω^Y, ω^X be two Kähler fibrations on f and on l respectively. We shall assume that ω^Y is the pull-back of ω^X so that the Kähler metric on Y is induced by the Kähler metric on X . Consider the following exact sequence

$$\bar{\mathcal{N}} : 0 \rightarrow \bar{T}f \rightarrow \bar{T}l|_Y \rightarrow \bar{N}_{X/Y} \rightarrow 0$$

where $N_{X/Y}$ is endowed with the quotient metric, we shall regard $\bar{\mathcal{N}}$ as an emi-1-cube of hermitian bundles on Y . Then the equivariant Todd form of $\bar{\mathcal{N}}$ has been defined in [T3] and it satisfies the identity

$$d\text{Td}_g(\bar{\mathcal{N}}) = \text{Td}_g(\bar{T}l|_Y, h^{Tl}) - \text{Td}_g(\bar{T}f, h^{Tf})\text{Td}_g(\bar{N}_{X/Y}).$$

We suppose that in the resolution $\bar{\xi}$, ξ_j are all l -acyclic and moreover η is f -acyclic. Then, by an easy argument of long exact sequence, we have the following exact sequence of hermitian vector bundles on S

$$\Xi : 0 \rightarrow l_*(\bar{\xi}_m) \rightarrow l_*(\bar{\xi}_{m-1}) \rightarrow \dots \rightarrow l_*(\bar{\xi}_0) \rightarrow f_* \bar{\eta} \rightarrow 0.$$

We may split Ξ into a family of short exact sequence of hermitian bundles from $j = 1$ to m

$$\chi_j : 0 \longrightarrow \text{Ker} d_j \longrightarrow \bar{\Xi}_j \xrightarrow{d_j} \text{Ker} d_{j-1} \longrightarrow 0$$

such that the kernel of every map d_{j-1} for $j = 2, \dots, m$ carries the metric induced by $\bar{\Xi}_j$ and $\text{Ker} d_0 = \bar{\Xi}_0 = f_* \bar{\eta}$, $\text{Ker} d_m = \bar{\Xi}_{m+1} = l_*(\bar{\xi}_m)$. We regard χ_j as a hermitian 1-cube on S and we set $\text{ch}_g(\bar{\Xi}) = \sum_{j=1}^m (-1)^j \text{ch}_g(\chi_j)$. Then it satisfies the differential equation

$$d\text{ch}_g(\bar{\Xi}) = \text{ch}_g(f_* \bar{\eta}) - \sum_{j=0}^m \text{ch}_g(l_*(\bar{\xi}_j)).$$

Theorem 4.4. (*Immersion formula*) *Let notations and assumptions be as above. Then the following identity holds in $\bigoplus_{p \geq 0} (D^{2p-1}(S, p)/\text{Im}d)$.*

$$\begin{aligned} \sum_{i=0}^m (-1)^i T_g(\omega^X, h^{\xi_i}) - T_g(\omega^Y, h^\eta) + \text{ch}_g(\Xi) &= -\frac{1}{(2\pi i)^{r_l}} \int_{X_{\mu_n}/S} \text{Td}_g(\overline{T}l) T_g(\bar{\xi}) \\ &- \frac{1}{(2\pi i)^{r_f}} \int_{Y_{\mu_n}/S} \text{Td}_g(\overline{N}) \text{Td}_g^{-1}(\overline{N}_{X/Y}) \text{ch}_g(\overline{\eta}) + \frac{1}{(2\pi i)^{r_f}} \int_{Y_{\mu_n}/S} \text{Td}_g(\overline{T}f) R_g(\overline{N}_{X/Y}) \text{ch}_g(\overline{\eta}) \end{aligned}$$

where r_f and r_l are the relative dimensions of Y_{μ_n}/S and of X_{μ_n}/S respectively.

Proof. This is the combination of [BM, Theorem 0.1 and 0.2], the main theorems in that paper. \square

Remark 4.5. Denote by $\Delta(f, l, i_* \overline{\eta}, \bar{\xi})$ the differential form which measures the difference

$$\begin{aligned} \sum_{i=0}^m (-1)^i T_g(\omega^X, h^{\xi_i}) - T_g(\omega^Y, h^\eta) + \text{ch}_g(\Xi) &+ \frac{1}{(2\pi i)^{r_l}} \int_{X_{\mu_n}/S} \text{Td}_g(\overline{T}l) T_g(\bar{\xi}) \\ &+ \frac{1}{(2\pi i)^{r_f}} \int_{Y_{\mu_n}/S} \text{Td}_g(\overline{N}) \text{Td}_g^{-1}(\overline{N}_{X/Y}) \text{ch}_g(\overline{\eta}) - \frac{1}{(2\pi i)^{r_f}} \int_{Y_{\mu_n}/S} \text{Td}_g(\overline{T}f) R_g(\overline{N}_{X/Y}) \text{ch}_g(\overline{\eta}) \end{aligned}$$

in Theorem 4.4. Let us go back to the same situation described before Lemma 2.8 and assume that the following diagrams

$$\begin{array}{ccc} Y \times Z & \xrightarrow{i_Z} & X \times Z \\ & \searrow f_Z \quad \swarrow l_Z & \\ & S \times Z & \end{array} \quad \text{and} \quad \begin{array}{ccc} Y \times Z_1 & \xrightarrow{i_{Z_1}} & X \times Z_1 \\ & \searrow f_{Z_1} \quad \swarrow l_{Z_1} & \\ & S \times Z_1 & \end{array}$$

are obtained by smooth base changes. Then $Y \times Z$ and $X \times Z_1$ intersect transversely along $Y \times Z_1$ and the singular currents can be pulled back. Moreover, similar to Lemma 2.8, the restriction of $\Delta(f_Z, l_Z, i_{Z*} \overline{\eta}, \bar{\xi})$ to $S \times Z_1$ is equal to the differential form $\Delta(f_{Z_1}, l_{Z_1}, i_{Z_1*} \overline{\eta} |_{Y \times Z_1}, \bar{\xi} |_{X \times Z_1})$.

Proposition 4.6. *Let Y be a μ_n -equivariant arithmetic scheme over (D, Σ, F_∞) and let \overline{N} be a μ_n -equivariant hermitian vector bundle on Y . Suppose that the μ_n -action on Y is trivial and consider the zero section embedding*

$$i : Y \hookrightarrow P := \mathbb{P}(N \oplus \mathcal{O}_Y)$$

with hermitian normal bundle \overline{N} and the natural projection $\pi : P \rightarrow Y$. Then for any element $x \in \widehat{K}_m(Y, \mu_n)_{\mathbb{Q}}$ with integer $m \geq 1$, the following identity

$$x - R_g(N) \cdot x = \pi_* i_*(x)$$

holds in $\widehat{K}_m(Y, \mu_n)_{\mathbb{Q}}$.

Proof. By the definition of the action of $R_g(N)$ on $\widehat{K}_m(Y, \mu_n)_{\mathbb{Q}}$, the map $x \mapsto x - R_g(N) \cdot x$ is defined via the chain homotopy

$$\Pi_k^{(0)}(\overline{E}) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(R_g(\overline{N}) \bullet \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2)$$

for the square

$$\begin{array}{ccc} \widetilde{\mathbb{Z}}C_*(Y, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(Y_{\mu_n}, p)_{R_n} \\ \downarrow \text{Id} & & \downarrow \text{Id} \\ \widetilde{\mathbb{Z}}C_*(Y, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(Y_{\mu_n}, p)_{R_n}. \end{array}$$

According to Proposition 4.3, to define the morphism $i_* : \widehat{K}_m(Y, \mu_n)_{\mathbb{Q}} \rightarrow \widehat{K}_m(P, \mu_n)_{\mathbb{Q}}$, we may choose a resolution F_\bullet of $i_*\mathcal{O}_Y$ on P such that every F_j is π -acyclic. We shall endow F_\bullet with the metrics satisfying the Bismut's assumption (A). Then we have an exact sequence of hermitian bundles on Y

$$\Xi : 0 \rightarrow \pi_*(\overline{F}_m) \rightarrow \pi_*(\overline{F}_{m-1}) \rightarrow \dots \rightarrow \pi_*(\overline{F}_0) \rightarrow \overline{\mathcal{O}}_Y \rightarrow 0.$$

Like before, splitting Ξ into a family of short exact sequence of hermitian bundles from $j = 1$ to m

$$\chi_j : 0 \longrightarrow \text{Ker} d_j \longrightarrow \Xi_j \xrightarrow{d_j} \text{Ker} d_{j-1} \longrightarrow 0,$$

we may construct a chain homotopy

$$H_{\pi \circ i}(\overline{E}) := \sum_{j=1}^m (-1)^j \chi_j \otimes \overline{E} \in \widetilde{\mathbb{Z}}C_{k+1}(Y, \mu_n)$$

between the maps Id and $\pi_* \circ i_* : \widetilde{\mathbb{Z}}C_*(Y, \mu_n) \rightarrow \widetilde{\mathbb{Z}}C_*(Y, \mu_n)$. Consequently, the formula

$$\mathbf{H}_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(H_{\pi \circ i}(\overline{E}))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2)$$

decides a chain homotopy between $\text{ch}_g \circ \text{Id}$ and $\text{ch}_g \circ \pi_* \circ i_*$. Then $\mathbf{H}_k^{(1)} + \Pi_k^\pi \circ i_* + \pi_{\mu_n} \circ (\text{Td}_g(\overline{T}\pi) \bullet \mathbf{H}_k)$ also decides a chain homotopy between $\text{ch}_g \circ \text{Id}$ and $\text{Id} \circ \text{ch}_g$. We compare it with $\Pi_k^{(0)}$.

Firstly, denote by Pr_P (resp. Pr_Y) the natural projection from $P \times (\mathbb{P}^1)^k$ (resp. $Y \times (\mathbb{P}^1)^k$) to P (resp. Y). Then, according to the functoriality of projective space bundle construction we have used before, $\text{Pr}_P^* \overline{F}_\bullet$ provides a resolution of $i_* \overline{\mathcal{O}}_{Y \times (\mathbb{P}^1)^k}$ on $P \times (\mathbb{P}^1)^k$. Hence we have an exact sequence

$$\Xi' : 0 \rightarrow \pi_*(\text{Pr}_P^* \overline{F}_m) \rightarrow \pi_*(\text{Pr}_P^* \overline{F}_{m-1}) \rightarrow \dots \rightarrow \pi_*(\text{Pr}_P^* \overline{F}_0) \rightarrow \overline{\mathcal{O}}_{Y \times (\mathbb{P}^1)^k} \rightarrow 0$$

which can be split into a family of short exact sequence of hermitian bundles from $j = 1$ to m

$$\chi'_j : 0 \longrightarrow \text{Ker} d_j \longrightarrow \Xi'_j \xrightarrow{d_j} \text{Ker} d_{j-1} \longrightarrow 0.$$

Furthermore, the short exact sequence of hermitian 1-cube

$$H^{(j)}(\overline{E}) : 0 \longrightarrow \chi'_j \otimes \text{tr}_k \circ \lambda(\overline{E}) \xrightarrow{\text{Id}} \text{Pr}_Y^* \chi_j \otimes \text{tr}_k \circ \lambda(\overline{E}) \longrightarrow 0 \longrightarrow 0$$

forms a hermitian 2-cube on $Y \times (\mathbb{P}^1)^k$. We set

$$\begin{aligned} \widetilde{\mathbf{H}}_k(\overline{E}) := & \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}_g^0 \left(\sum_{j=1}^m (-1)^j \text{tr}_2 \circ \lambda(H^{(j)}(\overline{E})) \right) \wedge \\ & C_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2), \end{aligned}$$

it satisfies the differential equation

$$\begin{aligned} d \circ \widetilde{\mathbf{H}}_k(\overline{E}) = & \widetilde{\mathbf{H}}_k \circ d\overline{E} + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(H_{\pi \circ i}(\overline{E}))) \\ & \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ & - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0 \left(\sum_{j=1}^m (-1)^j \text{tr}_1 \circ \lambda(\chi'_j) \boxtimes \text{tr}_k \circ \lambda(\overline{E}) \right) \\ & \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\ & + \sum_{j=0}^m (-1)^j \Pi_k'^\pi(\overline{F}_j \otimes \pi^* \overline{E}) \\ = & \widetilde{\mathbf{H}}_k \circ d\overline{E} + \mathbf{H}_k^{(1)}(\overline{E}) + \Pi_k'^\pi \circ i_*(\overline{E}) \\ & - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(\text{ch}_g(\Xi' \otimes \text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2). \end{aligned}$$

On the other hand, we apply the immersion formula to the resolution $\text{Pr}_P^* \overline{F} \cdot \otimes \text{tr}_k \circ \lambda(\overline{E})$. We then have

$$\begin{aligned} \Pi_k''^\pi \circ i_*(\overline{E}) = & - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(\text{ch}_g(\Xi' \otimes \text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2) \\ & - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\frac{1}{(2\pi i)^{r_\pi}} \int_{P_{\mu_n}/Y} \text{Td}_g(\overline{T\pi}) T_g(\text{Pr}_P^* \overline{F} \cdot \otimes \text{tr}_k \circ \lambda(\overline{E})), \right. \\ & \left. \log |z_1|^2, \dots, \log |z_k|^2 \right) \\ & + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(R_g(\overline{N}) \bullet \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2) \end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d\Delta(\mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2) \\
& = - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(\mathrm{ch}_g(\Xi' \otimes \mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2) \\
& \quad - \pi_{\mu_n*} \circ (\mathrm{Td}_g(\overline{T\pi}) \bullet \mathbf{H}_k(\overline{E})) + \Pi_k^{(0)}(\overline{E}) \\
& \quad + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d\Delta(\mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2).
\end{aligned}$$

We then formally define a product $C_{k+1}(\Delta(\mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2)$ in the same way as (9), and we set

$$\Delta_k(\overline{E}) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(\Delta(\mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2).$$

Again, it is readily checked by Lemma 4.5 that

$$\begin{aligned}
& \Delta_{k-1}(d\overline{E}) - d\Delta_k(\overline{E}) \\
& = \frac{(-1)^k}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d\Delta(\mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2).
\end{aligned}$$

Getting together all the above discussions, we see that $\widetilde{\mathbf{H}}_k + \Delta_k$ provides a homotopy between $\Pi_k^{(0)}$ and $\mathbf{H}_k^{(1)} + \Pi_k^\pi \circ i_* + \pi_{\mu_n*} \circ (\mathrm{Td}_g(\overline{T\pi}) \bullet \mathbf{H}_k)$ which implies that $x - R_g(N) \cdot x = \pi_* i_*(x)$ for any element $x \in \widehat{K}_m(Y, \mu_n)_\mathbb{Q}$ with integer $m \geq 1$. \square

Corollary 4.7. *Let S be another μ_n -equivariant arithmetic scheme with the trivial μ_n -action. Let $f : Y \rightarrow S$ and $g = f \circ \pi : P \rightarrow S$ be two equivariant and proper morphisms which are smooth on the generic fibres. Then the identity*

$$f_*(x) - f_*(R_g(N) \cdot x) = g_* \circ i_*(x)$$

holds in $\widehat{K}_m(S, \mu_n)_\mathbb{Q}$ for any element $x \in \widehat{K}_m(Y, \mu_n)$.

Proof. This is an immediate consequence of Proposition 4.6 and Corollary 3.6. \square

Now, we consider general situation. Let X, S be two μ_n -equivariant arithmetic schemes over (D, Σ, F_∞) , and let Y be a μ_n -equivariant arithmetic closed subscheme of X with immersion $i : Y \rightarrow X$. Let $g : X \rightarrow S$ and $f = g \circ i : Y \rightarrow S$ be two equivariant and proper morphisms which are smooth on the generic fibres. We shall suppose that the μ_n -actions on Y and on S are trivial (e.g. $Y = X_{\mu_n}, S = \mathrm{Spec} D$). Then the main result in this subsection is the following.

Theorem 4.8. *For any element $x \in \widehat{K}_m(Y, \mu_n)$ with integer $m \geq 1$, the identity*

$$f_*(x) - f_*(R_g(N_{X/Y}) \cdot x) = g_* \circ i_*(x)$$

holds in $\widehat{K}_m(S, \mu_n)_\mathbb{Q}$.

To prove Theorem 4.8, we use the deformation to the normal cone construction. Denote by W the blowing up of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, and denote by $q_W : W \rightarrow \mathbb{P}^1$ the composition of the blow-down map $W \rightarrow X \times \mathbb{P}^1$ with the natural projection $X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Then we have

$$q_W^{-1}(t) \cong \begin{cases} X \times \{t\}, & \text{if } t \neq 0, \\ P \cup \tilde{X}, & \text{if } t = 0, \end{cases}$$

where \tilde{X} is isomorphic to the blowing up of X along Y and P is the projective space bundle $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$. Let $j : Y \times \mathbb{P}^1 \rightarrow W$ be the canonical closed immersion induced by $i \times \text{Id}$, then the component \tilde{X} doesn't meet $j(Y \times \mathbb{P}^1)$ and the intersection of $j(Y \times \mathbb{P}^1)$ with P is exactly the image of Y under the zero section embedding. Moreover, denote by s_t the canonical section $Y \cong Y \times \{t\} \hookrightarrow Y \times \mathbb{P}^1$ for every t and denote by u_t the natural inclusion $q_W^{-1}(t) \hookrightarrow W$. We have two Tor-independent squares

$$\begin{array}{ccc} Y \times \mathbb{P}^1 & \xrightarrow{j} & W \\ s_t \uparrow & & \uparrow u_t \\ Y & \xrightarrow{i} & X \end{array}$$

with $t \neq 0$ and

$$\begin{array}{ccc} Y \times \mathbb{P}^1 & \xrightarrow{j} & W \\ s_0 \uparrow & & \uparrow u_0 \\ Y & \xrightarrow{i_0} & \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y). \end{array}$$

Notice that the complement $X \setminus Y$ is contained in $W \setminus Y \times \mathbb{P}^1$, we have natural pull-back morphism $u_t^* : \widehat{K}_{Y \times \mathbb{P}^1, m}(W, \mu_n) \rightarrow \widehat{K}_{Y, m}(X, \mu_n)$.

Lemma 4.9. *For any $t \neq 0$, the diagram*

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow[\cong]{j_*} & \widehat{K}_{Y \times \mathbb{P}^1, m}(W, \mu_n) \\ \downarrow s_t^* & & \downarrow u_t^* \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow[\cong]{i_*} & \widehat{K}_{Y, m}(X, \mu_n) \end{array}$$

is commutative.

Proof. Write $c_t^* : \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) \rightarrow \widehat{K}_m(Y, \mu_n)$ for the composition $i_*^{-1} \circ u_t^* \circ j_*$. We need to show that $c_t^* = s_t^*$. The morphism s_t^* is deduced from the commutativity between s_t^* and $\widehat{\text{ch}}_g$, while the morphism c_t^* is deduced from the homotopy defining j_* and the homotopy defining i_* . Since the K-theory and the Deligne-Beilinson cohomology are both \mathbb{A}^1 -homotopy invariant and s_t are sections of the natural projection $Y \times \mathbb{P}^1 \rightarrow Y$, the statement in this lemma will follow.

from the commutativity of the diagram

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow[\cong]{j_{0*}} & \widehat{K}_{Y \times \mathbb{P}^1, m}(P', \mu_n) \\ \downarrow s_t^* & & \downarrow u_t^* \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow[\cong]{i_{0*}} & \widehat{K}_{Y, m}(P, \mu_n) \end{array} \quad (13)$$

where $P' = \mathbb{P}((N_{X/Y} \boxtimes \mathcal{O}(-1)) \oplus \mathcal{O}_{Y \times \mathbb{P}^1})$ is the projective completion of $N_{W/Y \times \mathbb{P}^1}$ over $Y \times \mathbb{P}^1$. It is equivalent to show that the following diagram

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow{j_{0*}} & \widehat{K}_m(P', \mu_n) \\ \downarrow s_t^* & & \downarrow u_t^* \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow{i_{0*}} & \widehat{K}_m(P, \mu_n) \end{array} \quad (14)$$

is commutative because the morphism $i_{0*} : \widehat{K}_m(Y, \mu_n) \rightarrow \widehat{K}_m(P, \mu_n)$ is injective. We endow $N_{X/Y} \boxtimes \mathcal{O}(-1)$ with the product metric coming from the metric on $N_{X/Y}$ and the Fubini-Study metric on $\mathcal{O}(-1)$, then the pull-back of $\overline{N}_{W/Y \times \mathbb{P}^1}$ along s_t is isometric to $\overline{N}_{X/Y}$ so that the pull-back along s_t of the Koszul resolution and of the corresponding Bott-Chern singular current with respect to j_0 is exactly the Koszul resolution and the corresponding Bott-Chern singular current with respect to i_0 . According to the construction of the homotopies defining j_{0*} and i_{0*} , we get the commutativity of the diagram (14) and hence of (13). So we are done. \square

Corollary 4.10. *For any $t \neq 0$, the diagram*

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow{j_*} & \widehat{K}_m(W, \mu_n) \\ \downarrow s_t^* & & \downarrow u_t^* \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow{i_*} & \widehat{K}_m(X, \mu_n) \end{array}$$

is commutative.

Remark 4.11. Using the same argument as in Lemma 4.9, we know that the diagram

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow{j_*} & \widehat{K}_m(W, \mu_n) \\ \downarrow s_0^* & & \downarrow u_0^* \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow{i_{0*}} & \widehat{K}_m(P, \mu_n) \end{array}$$

is also commutative.

Next, we consider the commutative diagram

$$\begin{array}{ccc} & W & \\ u_t \uparrow & \searrow g & \\ X & \xrightarrow{f} & S \end{array}$$

with $t \neq 0$ and we compare the map $f_* \circ u_t^*$ with the map g_* from $\widehat{K}_m(W, \mu_n)_{\mathbb{Q}}$ to $\widehat{K}_m(S, \mu_n)_{\mathbb{Q}}$.

Firstly, for any μ_n -invariant Kähler metric ω^X on X which induces an invariant Kähler metric ω^Y on Y , there exists a μ_n -invariant Kähler metric ω^W on W such that the restrictions of ω^W to $X \cong X \times \{t\}$ with $t \neq 0$ and to $Y \cong Y \times \{0\}$ are exactly ω^X and ω^Y . This fact follows from [T2, Lemma 3.5]. Actually, such a metric is constructed via the Grassmannian graph construction. In this construction, we have an embedding $W \rightarrow X \times \mathbb{P}^r \times \mathbb{P}^1$ and the metric ω^W is the μ_n -average of the restriction of a product metric on $X \times \mathbb{P}^r \times \mathbb{P}^1$. We fix such an invariant Kähler metric ω^W on W and endow all submanifolds of W with the induced metrics. Moreover, all normal bundles appearing in the construction of the deformation to the normal cone will be endowed with the quotient metrics.

Secondly, to the three divisors $u_t(X)$, $u_0(P)$ and $u_0(\tilde{X})$ in W , we have the following result.

Lemma 4.12. *Over W , there are μ_n -invariant hermitian metrics on $\mathcal{O}(X)$, $\mathcal{O}(P)$ and $\mathcal{O}(\tilde{X})$ such that the isometry $\overline{\mathcal{O}}(X) \cong \overline{\mathcal{O}}(P) \otimes \overline{\mathcal{O}}(\tilde{X})$ holds and such that the restriction of $\overline{\mathcal{O}}(X)$ to X yields the metric of $N_{W/X}$, the restriction of $\overline{\mathcal{O}}(\tilde{X})$ to \tilde{X} yields the metric of $N_{W/\tilde{X}}$ and the restriction of $\overline{\mathcal{O}}(P)$ to P yields the metric of $N_{W/P}$.*

Proof. choose metric on $\mathcal{O}(P)$ in a small neighborhood of P such that the restriction of $\overline{\mathcal{O}}(P)$ to P yields the metric of the normal bundle. Do the same for $\mathcal{O}(\tilde{X})$. Since X is closed and disjoint from \tilde{X} and P , we can extend these metrics via a partition of unity to metrics defined on W so that the restriction of the metric that $\mathcal{O}(X)$ inherits from the isomorphism $\mathcal{O}(X) \cong \mathcal{O}(P) \otimes \mathcal{O}(\tilde{X})$ yields the metric of the normal bundle $N_{W/X}$. We then take the μ_n -averages of these metrics to make them μ_n -invariant. Since the metrics on $N_{W/X}$, $N_{W/P}$ and $N_{W/\tilde{X}}$ are already μ_n -invariant, the μ_n -invariant metrics on $\mathcal{O}(X)$, $\mathcal{O}(P)$ and $\mathcal{O}(\tilde{X})$ obtained as above have the properties that we require. \square

Now, consider the canonical Koszul resolution

$$0 \rightarrow \overline{\mathcal{O}}(-X) \rightarrow \overline{\mathcal{O}}_W \rightarrow u_{t*} \overline{\mathcal{O}}_X \rightarrow 0.$$

The associated equivariant singular Bott-Chern current $T_g(W/X)$ satisfies the identity

$$dT_g(W/X) = \text{ch}_g^0(\overline{\mathcal{O}}_W) - \text{ch}_g^0(\overline{\mathcal{O}}(-X)) - u_{t*}[\text{ch}_g^0(\overline{\mathcal{O}}_X) \text{Td}_g^{-1}(\overline{N}_{W/X})].$$

We claim the following result.

Lemma 4.13. *For any element $x \in \widehat{K}_m(W, \mu_n)_{\mathbb{Q}}$ with integer $m \geq 1$, the identity*

$$f_* \circ u_t^*(x) - f_*(R_g(N_{W/X}) \cdot u_t^*x) = g_*(x) - g_*(\overline{\mathcal{O}}(-X) \otimes x)$$

hold in $\widehat{K}_m(S, \mu_n)_{\mathbb{Q}}$.

Proof. Let \overline{E} be a g -acyclic hermitian k -cube in $\widehat{\mathcal{P}}(W, \mu_n)$. Since W admits a very ample invertible μ_n -sheaf which is relative to the morphism $g : W \rightarrow S$ (cf. [T2, Lemma 3.9]), we may assume that $\overline{\mathcal{O}}(-X) \otimes \overline{E}$ is also g -acyclic and $u_t^*\overline{E}$ is f -acyclic. Then we have a short exact sequence of hermitian k -cubes in $\widehat{\mathcal{P}}(S, \mu_n)$

$$\chi(\overline{E}) : 0 \rightarrow g_*(\overline{\mathcal{O}}(-X) \otimes \overline{E}) \rightarrow g_*(\overline{E}) \rightarrow f_*(u_t^*\overline{E}) \rightarrow 0,$$

which will be regarded as a hermitian $(k+1)$ -cube and as a chain homotopy between the maps $g_* - g_*(\overline{\mathcal{O}}(-X) \otimes)$ and $f_* \circ u_t^*$. Consequently, the formula

$$\mathbf{H}_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(\chi(\overline{E}))) \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2)$$

decides a chain homotopy between $\text{ch}_g \circ g_* - \text{ch}_g \circ g_*(\overline{\mathcal{O}}(-X) \otimes)$ and $\text{ch}_g \circ f_* \circ u_t^*$.

On the other hand, for any element $\alpha \in \bigoplus_{p \geq 0} D^{2p-*}(W_{\mu_n}, p)_{R_n}$, the formula

$$\frac{1}{(2\pi i)^{r_g}} \int_{W_{\mu_n}/S} T_g(W/X) \bullet \text{Td}_g(\overline{Tg}) \bullet \alpha + \frac{1}{(2\pi i)^{r_f}} \int_{X_{\mu_n}/S} \text{Td}_g(\overline{\mathcal{N}}) \bullet \text{Td}_g^{-1}(\overline{\mathcal{N}}_{W/X}) \bullet \alpha$$

gives a chain homotopy between the maps $g_{\mu_n!} \circ (\text{Td}_g(\overline{Tg}) \bullet) - g_{\mu_n!} \circ (\text{Td}_g(\overline{Tg}) \text{ch}_g^0(\overline{\mathcal{O}}(-X)) \bullet)$ and $f_{\mu_n!} \circ (\text{Td}_g(\overline{Tf}) \bullet u_t^*)$. Hence, it decides a chain homotopy between $g_{\mu_n!} \circ (\text{Td}_g(\overline{Tg}) \bullet \text{ch}_g) - g_{\mu_n!} \circ (\text{Td}_g(\overline{Tg}) \text{ch}_g^0(\overline{\mathcal{O}}(-X)) \bullet \text{ch}_g)$ and $f_{\mu_n!} \circ (\text{Td}_g(\overline{Tf}) \bullet u_t^* \circ \text{ch}_g)$. Like before, using the projection formula and the fact that the deformation to the normal cone construction is base-change invariant along smooth morphisms, we write the deduced homotopy as

$$\begin{aligned} \mathbf{H}_k^{(2)}(\overline{E}) &= \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left[\frac{1}{(2\pi i)^{r_g}} \int_{W_{\mu_n} \times (\mathbb{P}^1)^k / S \times (\mathbb{P}^1)^k} T_g(W/X) \bullet \text{Td}_g(\overline{Tg}) \text{ch}_g^0(\text{tr}_k \circ \lambda(\overline{E})) \right] \\ &\quad \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2) \\ &\quad + \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left[\frac{1}{(2\pi i)^{r_f}} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / S \times (\mathbb{P}^1)^k} \text{Td}_g(\overline{\mathcal{N}}) \bullet \text{Td}_g^{-1}(\overline{\mathcal{N}}_{W/X}) \right. \\ &\quad \left. \text{ch}_g^0(\text{tr}_k \circ \lambda(u_t^*\overline{E})) \right] \wedge C_k(\log |z_1|^2, \dots, \log |z_k|^2). \end{aligned}$$

Now, we denote by $H_\chi(\overline{E})$ the following 2-cube of hermitian bundles on $S \times (\mathbb{P}^1)^k$

$$\begin{array}{ccccc}
g_*(\mathrm{tr}_k \circ \lambda(\overline{\mathcal{O}}(-X) \otimes \overline{E})) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda(g_*(\overline{\mathcal{O}}(-X) \otimes \overline{E})) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
g_*(\mathrm{tr}_k \circ \lambda(\overline{E})) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda(g_*(\overline{E})) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
f_*(\mathrm{tr}_k \circ \lambda(u_t^* \overline{E})) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda(f_*(u_t^* \overline{E})) & \longrightarrow & 0
\end{array}$$

and we set

$$\widetilde{\mathbf{H}}_k(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0(\mathrm{tr}_2 \circ \lambda(H_\chi(\overline{E}))) \wedge C_{k+2}(\log |z_1|^2, \dots, \log |z_{k+2}|^2),$$

it satisfies the differential equation

$$\begin{aligned}
d \circ \widetilde{\mathbf{H}}_k(\overline{E}) &= \widetilde{\mathbf{H}}_k \circ d\overline{E} + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0(\mathrm{tr}_{k+1} \circ \lambda(\chi(\overline{E}))) \\
&\quad \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
&\quad - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0(\mathrm{tr}_1 \circ \lambda(\chi(\mathrm{tr}_k \circ \lambda(\overline{E})))) \\
&\quad \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2) \\
&\quad - \Pi_k'^g(\overline{E}) + \Pi_k'^g(\overline{\mathcal{O}}(-X) \otimes \overline{E}) + \Pi_k'^f(u_t^* \overline{E}) \\
&= \widetilde{\mathbf{H}}_k \circ d\overline{E} + \mathbf{H}_k^{(1)}(\overline{E}) - \Pi_k'^g(\overline{E}) + \Pi_k'^g(\overline{\mathcal{O}}(-X) \otimes \overline{E}) + \Pi_k'^f(u_t^* \overline{E}) \\
&\quad - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0(\mathrm{tr}_1 \circ \lambda(\chi(\mathrm{tr}_k \circ \lambda(\overline{E})))) \\
&\quad \wedge C_{k+1}(\log |z_1|^2, \dots, \log |z_{k+1}|^2).
\end{aligned}$$

Similar to the tricks that we used frequently before, we set

$$\begin{aligned}
\mathbf{H}_k^{(2')}(\overline{E}) &= \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\frac{1}{(2\pi i)^{r_g}} \int_{W_{\mu_n} \times (\mathbb{P}^1)^k / S \times (\mathbb{P}^1)^k} T_g(W/X) \bullet \mathrm{Td}_g(\overline{Tg}) \right. \\
&\quad \left. \mathrm{ch}_g^0(\mathrm{tr}_k \circ \lambda(\overline{E})), \log |z_1|^2, \dots, \log |z_k|^2 \right) \\
&\quad + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\frac{1}{(2\pi i)^{r_f}} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / S \times (\mathbb{P}^1)^k} \mathrm{Td}_g(\overline{\mathcal{N}}) \bullet \mathrm{Td}_g^{-1}(\overline{\mathcal{N}}_{W/X}) \right. \\
&\quad \left. \mathrm{ch}_g^0(\mathrm{tr}_k \circ \lambda(u_t^* \overline{E})), \log |z_1|^2, \dots, \log |z_k|^2 \right).
\end{aligned}$$

then our lemma follows from the Bimut-Ma's immersion formula and the fact that there exists a natural homotopy between $\mathbf{H}_k^{(2')}(\overline{E})$ and $\mathbf{H}_k^{(2)}(\overline{E})$. So we are done. \square

Remark 4.14. Similar to Lemma 4.13, we consider other three divisors $W \xleftarrow{u_0} P \xrightarrow{p} S$,

$W \xleftarrow{u_0} \tilde{X} \xrightarrow{h_1} S$ and $W \xleftarrow{u_0} P \cap \tilde{X} \xrightarrow{h_2} S$ and corresponding Koszul resolutions

$$0 \rightarrow \overline{\mathcal{O}}(-P) \rightarrow \overline{\mathcal{O}}_W \rightarrow u_{0*}\overline{\mathcal{O}}_P \rightarrow 0,$$

$$0 \rightarrow \overline{\mathcal{O}}(-\tilde{X}) \rightarrow \overline{\mathcal{O}}_W \rightarrow u_{0*}\overline{\mathcal{O}}_{\tilde{X}} \rightarrow 0,$$

and

$$0 \rightarrow \overline{\mathcal{O}}(-\tilde{X}) \otimes \overline{\mathcal{O}}(-P) \rightarrow \overline{\mathcal{O}}(-\tilde{X}) \oplus \overline{\mathcal{O}}(-P) \rightarrow \overline{\mathcal{O}}_W \rightarrow u_{0*}\overline{\mathcal{O}}_{\tilde{X} \cap P} \rightarrow 0.$$

Then, for any element $x \in \hat{K}_m(W, \mu_n)_{\mathbb{Q}}$, we have

$$p_* \circ u_0^*(x) - p_*(R_g(N_{W/P}) \cdot u_0^*x) = g_*(x) - g_*(\overline{\mathcal{O}}(-P) \otimes x),$$

$$h_{1*} \circ u_0^*(x) - h_{1*}(R_g(N_{W/\tilde{X}}) \cdot u_0^*x) = g_*(x) - g_*(\overline{\mathcal{O}}(-\tilde{X}) \otimes x),$$

and

$$\begin{aligned} h_{2*} \circ u_0^*(x) - h_{2*}(R_g(N_{W/P \cap \tilde{X}}) \cdot u_0^*x) &= g_*(x) - g_*(\overline{\mathcal{O}}(-P) \otimes x) - g_*(\overline{\mathcal{O}}(-\tilde{X}) \otimes x) \\ &\quad + g_*(\overline{\mathcal{O}}(-P) \otimes \overline{\mathcal{O}}(-\tilde{X}) \otimes x) \end{aligned}$$

which hold in $\hat{K}_m(S, \mu_n)_{\mathbb{Q}}$.

Now, we are ready to give the proof of Theorem 4.8.

Proof. (of Theorem 4.8) Let x be an element in $\hat{K}_m(Y, \mu_n)_{\mathbb{Q}}$, we consider the following two diagrams

$$\begin{array}{ccccc} Y \times \mathbb{P}^1 & \xrightarrow{j} & W & & \\ s_t \uparrow & & \uparrow u_t & \searrow h & \\ Y & \xrightarrow{i} & X & \xrightarrow{g} & S \end{array}$$

with $t \neq 0$ and

$$\begin{array}{ccccc} Y \times \mathbb{P}^1 & \xrightarrow{j} & W & & \\ s_0 \uparrow & & \uparrow u_0 & \searrow h & \\ Y & \xrightarrow{i_0} & \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y) & \xrightarrow{p} & S. \end{array}$$

By Corollary 4.10 and the fact that s_t is a section of the natural projection Pr from $Y \times \mathbb{P}^1$ to Y , we have that $i_*(x) = u_t^* \circ j_* \circ Pr^*(x)$ and hence $g_* \circ i_*(x) = g_* \circ u_t^* \circ j_* \circ Pr^*(x)$. According to Lemma 4.13,

$$g_* \circ u_t^*(j_* Pr^*x) = h_*(j_* Pr^*x) - h_*(\overline{\mathcal{O}}(-X) \otimes j_* Pr^*x) + g_*(R_g(N_{W/X}) \cdot i_*x).$$

Similarly, we have

$$g_* \circ u_0^*(j_* Pr^* x) = h_*(j_* Pr^* x) - h_*(\overline{O}(-P) \otimes j_* Pr^* x) + p_*(R_g(N_{W/P}) \cdot i_* x).$$

Notice that the image $j(Y \times \mathbb{P}^1)$ doesn't meet \tilde{X} , the localization sequence of the higher equivariant arithmetic K-groups implies that $u_0^*(j_* Pr^* x)$ vanishes in $\hat{K}_m(\tilde{X}, \mu_n)_{\mathbb{Q}}$ and in $\hat{K}_m(P \cap \tilde{X}, \mu_n)_{\mathbb{Q}}$ so that

$$h_*(\overline{O}(-X) \otimes j_* Pr^* x) = h_*(\overline{O}(-P) \otimes j_* Pr^* x).$$

This can be seen from the several identities mentioned in Remark 4.14. On the other hand,

$$\begin{aligned} R_g(N_{W/X}) \cdot i_* x &= R_g(N_{W/X}) \text{ch}_g(i_* x) = R_g(N_{W/X}) i_*(\text{Td}_g^{-1}(N_{X/Y}) \text{ch}_g(x)) \\ &= i_*(i^* R_g(N_{W/X}) \text{Td}_g^{-1}(N_{X/Y}) \text{ch}_g(x)) = 0. \end{aligned}$$

The same reasoning gives that $R_g(N_{W/P}) \cdot i_* x = 0$ also. So $g_* \circ i_*(x)$ is actually equal to $p_* \circ i_{0*}(x)$. Therefore, the statement in Theorem 4.8 follows from Corollary 4.7. \square

4.3 Proof of the statement

In this subsection, we give a complete proof of Theorem 4.2. Denote by i the closed immersion $X_{\mu_n} \rightarrow X$, then the arithmetic concentration theorem (cf. [T3, Theorem 5.2]) tells us that

$$i_* : \hat{K}_m(X_{\mu_n}, \mu_n)_{\rho} \cong \hat{K}_m(X, \mu_n)_{\rho}$$

with inverse map $\otimes \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}) \circ \tau$.

Then let x be any element in $\hat{K}_m(X, \mu_n)$, we apply Theorem 4.8 to the morphisms i, f and $f_{\mu_n} = f \circ i$ and we compute

$$\begin{aligned} f_*(x) &= f_*(i_* \circ \otimes \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}) \circ \tau(x)) \\ &= f_* \circ i_* (\otimes \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}) \circ \tau(x)) \\ &= f_{\mu_n*} (\otimes \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}) \circ \tau(x)) - f_{\mu_n*} (\otimes R_g(N_{X/X_{\mu_n}}) \circ \otimes \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}) \circ \tau(x)) \\ &= f_{\mu_n*} (\Lambda_R \circ \tau(x)) \end{aligned}$$

which holds in $\hat{K}_m(Y, \mu_n)_{\rho} \otimes \mathbb{Q}$. This completes the proof of Theorem 4.2.

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